



On the Computation of the Lagrange Multiplier for the Variational Iteration Method (VIM) for Solving Differential Equations

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Authors' contributions

This work was carried out with participation from all authors. Author FO suggested the study. Author NO designed and wrote the manuscript under the supervision of authors FO and MOD. All authors read and approved the final manuscript.

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Abstract

In this study, the Variational Iteration Method (VIM) is applied in finding the solution of differential equations with emphasis laid on the choice of the Lagrange multiplier used while employing VIM. Building on existing methods and variational theories, the operator D-Method and integrating factor are employed in certain aspects in the determination of exact Lagrange multiplier for VIM. When results of the computed exact Lagrange multiplier were compared with results of approximate Lagrange multiplier, it was observed that the computed exact Lagrange multiplier reduced significantly the number of iterations required to get a good approximate result, and in some cases the result converged to the exact solution after a single iteration. Evaluations are carried out using MAPLE Software.

Keywords: Integrating factor; operator D method; variational iteration method.

1 Introduction

Over the years different analytical and numerical schemes have been developed for solving differential equations owing to the important role played by differential equation in modelling real life phenomena's and

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problems arising in physics, engineering, economics etc. Schemes for solving differential equations include, the Finite Difference method (FDM), the Adomian decomposition method (ADM), Homotopy perturbation method, the Elzaki transformation method etc. One such scheme that has gained lots of attention in recent times is the Variational Iteration Method (VIM). The method was developed by Chinese mathematician Ji Huan He in 1997 [1]. Various authors have successfully applied the method to a range of problems owing to the flexibility, versatility and ease of implementation of the method once the Lagrange multiplier has been determined. Accurate determination of the Lagrange multiplier for the scheme is often not easy. As Mehmood et al. [2] puts it, the basic problem in the implementation of VIM is the exact identification of Lagrange multiplier which is a real hard task. Wu in [3] pointed out that the determination of the Lagrange multiplier for VIM for certain problems can be complex if not impossible. Over the years however, a number of authors have proposed various method for the computation of the Lagrange multiplier for VIM. He, Wu [4] in their summary of some iteration formula for VIM, presented a list of forms the Lagrange multiplier could take for different forms of problems one of which culminates into the now very popular formula

$$\lambda(s) = \frac{(-1)^n}{(n-1)!} (s-x)^{n-1} \quad \text{where } n \text{ is the order of the derivative.} \tag{1}$$

which abounds in literatures. Authors in [3] and [5] suggested a universal way to identify the Lagrange multiplier for VIM by implementing Laplace transform. Altintan and Ugor [6] proposed a method that defines the Lagrange multipliers for VIM as Matrix-valued functions. Building on existing methods and variational theories for the determination of the Lagrange multiplier for VIM, in this paper we apply the operator D-Method and integrating factor to certain aspect of the determination of accurate Lagrange multiplier for VIM. We apply restricted variation only to the non-linear term in the correction functional.

2 Overview of Variational Iteration Method (VIM)

To illustrate the basic concept of the technique, we consider the following nonlinear differential equation:

$$Lx + Nx = g(t) \tag{2}$$

Where: L is a linear operator, N is a nonlinear operator and $g(t)$ is a known function. The variational iteration method presents a correction functional for equation (2) in the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds \tag{3}$$

Where λ is a general Lagrange’s multiplier, which may be a constant or a function, and may be identified optimally via variational theory [7]. The subscript n denotes the n th approximation. \tilde{u}_n is considered as a restricted variation [8], i.e. $\delta\tilde{u}_n = 0$. Generally, in applying VIM the following three steps are followed

1. Establishing the correction functional
2. Identifying the Lagrange multiplier
3. Determining the initial iteration.

The step 2 is very crucial [5].

Distinct form of the Lagrange multiplier $\lambda(s)$ for different orders of problems, culminating in the formula (1), have formally being derived in [9]. These forms are derived by taking the variation of (3) with respect to the independent variable u_n , to get

$$\delta u_{n+1} = \delta u_n + \delta \left(\int_0^x \lambda(s)(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds \right) \tag{4}$$

Since \tilde{u}_n is a restricted variation, it implies that $\delta\tilde{u}_n = 0$.

Therefore (3) become

$$\delta u_{n+1} = \delta u_n + \delta \left(\int_0^x \lambda(s) Lu_n(s) ds \right) \tag{5}$$

If $Lu_n(s) = u'_n(s)$ in (5), then we may write (5) as

$$\delta u_{n+1} = \delta u_n + \delta \left(\int_0^x \lambda(s) u'_n(s) ds \right) \tag{6}$$

Integrating the integral sign in (6) using integration by part and thereafter setting $\delta u_{n+1} = 0$ since the extremum condition of u_{n+1} requires that δu_{n+1} is stationary we get the system

$$\left. \begin{aligned} 1 + \lambda|_{s=x} &= 0 \\ \lambda'|_{s=x} &= 0 \end{aligned} \right\} \tag{7}$$

Solving the above system of equations, it is easy to see that

$$\lambda(s) = -1 \tag{8}$$

Similarly, if $Lu_n(s) = u''_n(s)$ in (5), the following system of equation is easily derived

$$\left. \begin{aligned} 1 - \lambda'|_{s=x} &= 0 \\ \lambda|_{s=x} &= 0 \\ \lambda'|_{s=x} &= 0 \end{aligned} \right\} \tag{9}$$

And the Lagrange multiplier solved for as

$$\lambda(s) = s - x \tag{10}$$

...and so on.

Generally, for n th order ODE

$$u^{(n)} + f(u, u', u'', \dots, u^{(n-1)}) = g(x), u(0) = \alpha_0, u'(0) = \alpha_1, \dots, u^{(n-1)}(0) = \alpha_{n-1} \tag{11}$$

The VIM correction functional takes the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s) (u_n^{(n)}(s) + f(u_n(s), u'_n(s), u''_n(s), \dots, u_n^{(n-1)}(s)) - g(s)) ds \tag{12}$$

With $\lambda = \frac{(-1)^n}{(n-1)!} (s-x)^{n-1}$

And the iteration formula takes the form

$$u_{n+1}(x) = u_n(x) + \frac{(-1)^n}{(n-1)!} \int_0^x (s-x)^{n-1} (u_n^{(n)}(s) + f(u_n(s), u'_n(s), u''_n(s), \dots, u_n^{(n-1)}(s)) - g(s)) ds \tag{13}$$

The zeroth approximation may preferably be selected as

$$u_{0(x)} = u(0) + u'(0)x + \frac{1}{2!} u''(0)x^2 + \dots + \frac{1}{n-1} u^{(n-1)}(0)x^{n-1} \tag{14}$$

Where n is the order of the ODE.

VIM admits the use of $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ [10].

The above can be extended to PDEs.

2.1 Integrating factor

Consider the linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad (15)$$

Where P & Q are functions of x (but not of y) or constants

We multiply both sides of (15) by $e^{\int P dx}$ to get

$$e^{\int P dx} \left(\frac{dy}{dx} + Py \right) = Q e^{\int P dx} \quad (16)$$

The required solution for (15) is written as

$$y \times [I \cdot F] = \int Q [I \cdot F] dx + c \quad (17)$$

and

$$I \cdot F = e^{\int P dx} \quad (18)$$

This is referred to as the integrating factor; see ref. [11].

2.2 Operator D-method

Consider the linear differential equation of the form

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y^{(1)} + a_n y = g \quad (19)$$

Where $a_k, k = 1, \dots, n$ are numbers, $y^{(k)} = \frac{d^k y}{dx^k}$, and $g = g(x)$ is a known function of x .

If we denote *derivative operator* $D = \frac{d}{dx}$, then we may write the differential equation (19) as:

$$p(D)y = (D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)y = g \quad (20)$$

We may obtain an auxiliary function $p(m) = 0$ for (19) from $p(D) = 0$ as

$$p(m) = m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0 \quad (21)$$

(20) and (21) can be used in determining the n real or complex roots of (19) and hence a general solution from which we can determine particular solutions for (19).

We extend these ideas to the accurate determination of the Lagrange multiplier for VIM in the following problems.

3 Numerical Applications

Problem 1:

Consider the differential equation

$$y'' + y - 4x^2 = 8 \quad (22)$$

$$y(0) = 0, y'(0) = 1 \text{ with exact solution } y = 4x^2 + \sin x$$

Solution:

A. Problem solved using an approximate Lagrange multiplier

We proceed by constructing the Correction functional for (22) as

$$y_{n+1}(x) = y_n(x, t) + \int_0^x \lambda(s)(y_n''(s) + y_n(s) - 4s^2 - 8) ds \quad (23)$$

Using (1) we determine the Lagrange multiplier as

$$\lambda(s) = (s - x) \quad (24)$$

So that we get the iteration formula

$$y_{n+1}(x) = y_n(x, t) + \int_0^x (s - x)(y_n''(s) + y_n(s) - 4s^2 - 8) ds \quad (25)$$

We determine the initial and subsequent approximations as

$$y_0(x) = x$$

$$y_1(x) = x + \frac{1}{3}x^4 - \frac{1}{6}x^3 + 4x^2$$

$$y_2 = x - \frac{1}{6}x^3 + 4x^2 - \frac{1}{90}x^6 + \frac{1}{120}x^5$$

$$y_3 = x - \frac{1}{6}x^3 + 4x^2 + \frac{1}{120}x^5 + \frac{1}{5040}x^8 - \frac{1}{5040}x^7$$

$$y_4 = x - \frac{1}{6}x^3 + 4x^2 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 - \frac{1}{453600}x^{10} + \frac{1}{362880}x^9$$

$$y_5 = x - \frac{1}{6}x^3 + 4x^2 - \frac{1}{39916800}x^{11} + \frac{1}{59875200}x^{12} - \frac{1}{5040}x^7 + \frac{1}{120}x^5$$

$$+ \frac{1}{362880}x^9$$

$$y_6 = \frac{1}{6227020800}x^{13} - \frac{1}{10897286400}x^{14} - \frac{1}{39916800}x^{11} + \frac{1}{362880}x^9 - \frac{1}{5040}x^7$$

$$+ \frac{1}{120}x^5 + 4x^2 - \frac{1}{6}x^3 + x$$

We compare results of the exact and numerical solutions in the Table 1 and Fig. 1.

B. Problem solved using a more exact Lagrange multiplier

We proceed as above, by constructing a correction functional for (22) as;

$$y_{n+1}(x) = y_n(x, t) + \int_0^x \lambda(s)(y_n''(s) + y_n(s) - 4s^2 - 8) ds \quad (26)$$

To determine the Lagrange multiplier $\lambda(s)$ we take variation of (26) with respect to the independent variable y_n , to get

$$\delta y_{n+1}(x) = \delta y_n(x, t) + \delta \int_0^x \lambda(s)(y_n''(s) + y_n(s)) ds \quad (27)$$

$$\delta y_{n+1}(x) = \delta y_n(x, t) + \delta \int_0^x \lambda(s)(y_n''(s)) ds + \delta \int_0^x \lambda(s)y_n(s) ds \quad (28)$$

Using integration by part, we get

$$\delta y_{n+1}(x) = \delta y_n(x, t) + \delta \lambda(s)y_n'(s) - \delta \lambda'(s)y_n(s) + \delta \int_0^x \lambda''(s)y_n(s) ds + \delta \int_0^x \lambda(s)y_n(s) ds \quad (29)$$

Which is same as

$$\delta y_{n+1}(x) = \delta y_n(s)(1 - \lambda')|_{s=x} + \delta y_n'(s)(\lambda(s))|_{s=x} + \int_0^x \delta y_n(s)(\lambda(s) + \lambda''(s))|_{s=x} \quad (30)$$

We make (30) stationary by setting $\delta y_{n+1}(x) = 0$ to get the following system

$$\begin{aligned} \delta y_n: 1 - \lambda'(s) &= 0 & (a) \\ \delta y_n': \lambda(s) &= 0 & (b) \\ \delta y_n: \lambda(s) + \lambda''(s) &= 0 & (c) \end{aligned}$$

We proceed by solving the systems (a)-(c) using Operator D method. We use (c) as the differential equation and (a) and (b) as the boundary condition. From (c) we may write

$D^2 \lambda + \lambda = 0$, so that $\lambda = 0$, and $D = \pm i$ so that the general solution for (c) becomes

$$\lambda(s) = C_1 \cos(s) + C_2 \sin(s) \quad (31)$$

Applying (b) and (a) on (31), we determine

$$C_1 = -\sin(x) \text{ and } C_2 = \cos(x) \quad (32)$$

Hence

$$\lambda(s) = \sin(s - x) \quad (33)$$

Substituting (33) into the correction functional (26), we get the iteration formula

$$y_{n+1}(x) = y_n(x, t) + \int_0^x \sin(s - x)(y_n''(s) + y_n(s) - 4s^2 - 8) ds \quad (34)$$

We determine the initial and subsequent approximations as

$$\begin{aligned} y_0(x) &= x \\ y_1(x) &= \sin(x) - 4x^2 \\ y_2(x) &= \sin(x) - 4x^2 \end{aligned}$$

We note that the solution using a more exact Lagrange multiplier converges to the exact solution after just one iteration.

Problem 2:

Consider the differential equation [12]

$$y''' + 3y'' + 3y' + y = 0 \quad (35)$$

$$y(0) = 7, y'(0) = -7, y''(0) = 11 \text{ And exact solution } y = 7e^{-x} + 2x^2e^{-x}$$

Solution:

A. Problem solved using an approximate Lagrange multiplier

We proceed by constructing the Correction functional for (35) as

$$y_{n+1}(x) = y_n(x, t) + \int_0^x \lambda(s)(y_n'''(s) + 3y_n''(s) + 3y_n'(s) + y_n(s)) ds \tag{36}$$

Using (1) we determine the Lagrange multiplier as

$$\lambda(s) = \frac{(s-x)^2}{2} \tag{37}$$

So that we get the iteration formula

$$y_{n+1}(x) = y_n(x, t) + \int_0^x \frac{(s-x)^2}{2}(y_n'''(s) + 3y_n''(s) + 3y_n'(s) + y_n(s)) ds \tag{38}$$

We determine the initial and subsequent approximations as

$$y_0 = 7 - 7x + \frac{11}{2}x^2$$

$$y_1 = 7 - 7x + \frac{11}{2}x^2 - \frac{11}{120}x^5 - \frac{13}{12}x^4 - \frac{19}{6}x^3$$

$$y_3 = 7 - 7x + \frac{11}{2}x^2 - \frac{1}{3628800}x^{11} - \frac{691}{40320}x^8 - \frac{223}{360}x^6 - \frac{793}{5040}x^7 - \frac{17}{18144}x^9$$

$$- \frac{23}{907200}x^{10} - \frac{19}{6}x^3 + \frac{31}{24}x^4 - \frac{47}{120}x^5$$

$$y_4 = 7 - \frac{19}{6}x^3 + \frac{31}{24}x^4 + \frac{181}{630}x^7 + \frac{941}{10080}x^8 + \frac{2449}{181440}x^9 + \frac{1943}{1814400}x^{10}$$

$$+ \frac{1987}{39916800}x^{11} + \frac{59}{43545600}x^{12} + \frac{5}{249080832}x^{13} - \frac{47}{120}x^5 + \frac{67}{720}x^6$$

$$+ \frac{1}{7925299200}x^{14} + \frac{11}{2}x^2 - 7x$$

...and so on

We present and compare exact and numerical results in Table 2 and Fig. 2.

B. Problem solved using a more exact Lagrange multiplier

We proceed by constructing the Correction functional for (35) as

$$y_{n+1}(x) = y_n(x, t) + \int_0^x \lambda(s)(y_n'''(s) + 3y_n''(s) + 3y_n'(s) + y_n(s)) ds \tag{39}$$

We take variation of (39) with respect to the independent variable y_n . Using integration by part on (39) and making the result stationary by setting $\delta y_{n+1}(x) = 0$, we get the following systems.

$$\delta y_n: \lambda(s) - 3\lambda'(s) + 3\lambda''(s) - \lambda'''(s) = 0 \tag{a}$$

$$\delta y_n: 1 + \lambda''(s) - 3\lambda'(s) + 3\lambda(s) = 0 \tag{b}$$

$$\delta y_n': -3\lambda'(s) + 3\lambda(s) = 0 \tag{c}$$

$$\delta y_n': \lambda(s) = 0 \quad (d)$$

We proceed by solving the systems (a)-(d) using Operator D method. We use (a) as the differential equation and (b) to (d) as the boundary condition. From (a) we may write

$$-D^3\lambda + 3D^2\lambda - 3D\lambda + \lambda = 0 \quad (40)$$

$$\Rightarrow D = 1 \text{ (Trice)} \quad \lambda = 0 \quad (41)$$

Which implies that

$$\lambda(s) = Ae^s + Bse^s + Cs^2e^s \quad (42)$$

Applying (b)-(d) on (42), we get the following system

$$Ae^x + Bxe^x + Cx^2e^x = 0 \quad (43)$$

$$2Ae^x - Be^x + 2Bxe^x - 2Cxe^x + 2Cx^2e^x = 0 \quad (44)$$

$$1 + Ae^x - Be^x + Bxe^x + Ce^x - 2Cxe^x + 2Cx^2e^x = 0 \quad (45)$$

Solving the system (43) - (45), it's easy to determine the Lagrange multiplier as

$$\lambda(s) = -\frac{1}{2} \frac{e^s(s-x)^2}{e^x} \quad (46)$$

So that we get the iteration formula

$$y_{n+1}(x) = y_n(x, t) + \int_0^x -\frac{1}{2} \frac{e^s(s-x)^2}{e^x} (y_n''''(s) + 3y_n'''(s) + 3y_n''(s) + y_n(s)) ds \quad (47)$$

We determine the initial and subsequent approximations as

$$y_0 = 7 - 7x + \frac{11}{2}x^2$$

$$y_1 = 7 - 7x + \frac{11}{2}x^2 - \frac{1}{2} (11x^2e^x - 14xe^x - 4x^2 + 14e^x - 14) e^{-x}$$

$$y_2 = 7 - 7x + \frac{11}{2}x^2 - \frac{1}{2} (11x^2e^x - 14xe^x - 4x^2 + 14e^x - 14) e^{-x}$$

We observe again the convergence after a single iteration using a more exact Lagrange multiplier. We compare this result to the exact result in the Table 2.

Problem 3:

Consider the differential equation

$$y'''' - 4y''' - y'' - 4y = -24e^{2x}, \quad (48)$$

$$y(0) = 1, y'(0) = 2, y''(0) = 4, y'''(0) = 8 \text{ And exact solution } y = e^{2x}$$

Solution:

A. Problem solved using an approximate Lagrange multiplier

Again, we proceed by constructing the Correction functional for (48) as

$$y_{n+1}(x) = y_n(x, t) + \int_0^x \lambda(s)(y_n''''(s) - 4y_n'''(s) - y_n''(s) - 4y_n(s) + 24e^{2s}) ds \quad (49)$$

Using (1) we determine the Lagrange multiplier as

$$\lambda(s) = \frac{(s-x)^3}{6} \quad (50)$$

And the iteration formula as

$$y_{n+1}(x) = y_n(x, t) + \int_0^x \frac{(s-x)^3}{6} (y_n''''(s) - 4y_n'''(s) - y_n''(s) - 4y_n(s) + 24e^{2s}) ds \quad (51)$$

We determine the initial and subsequent approximations as

$$\begin{aligned} y_0 &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 \\ y_1 &= \frac{5}{2} + 5x + 5x^2 + \frac{10}{3}x^3 + \frac{5}{3}x^4 + \frac{2}{15}x^5 + \frac{1}{45}x^6 + \frac{2}{315}x^7 - \frac{3}{2}e^{2x} \\ y_2 &= \frac{25}{4} + \frac{25}{2}x + \frac{25}{2}x^2 + \frac{25}{3}x^3 + \frac{25}{6}x^4 + \frac{5}{3}x^5 + \frac{1}{5}x^6 + \frac{2}{63}x^7 - \frac{21}{4}e^{2x} \\ &\quad + \frac{1}{311850}x^{11} + \frac{1}{56700}x^{10} + \frac{19}{2520}x^8 + \frac{1}{3780}x^9 \\ y_3 &= 3x + \frac{125}{4}x^2 + \frac{125}{6}x^3 + \frac{125}{8} + \frac{1}{2554051500}x^{15} + \frac{1}{340540200}x^{14} \\ &\quad + \frac{1}{12162150}x^{13} + \frac{1}{267300}x^{12} + \frac{31}{1247400}x^{11} + \frac{79}{226800}x^{10} + \frac{37}{1260}x^8 \\ &\quad + \frac{17}{2835}x^9 - \frac{117}{8}e^{2x} + \frac{125}{12}x^4 + \frac{25}{6}x^5 + \frac{25}{18}x^6 + \frac{61}{315}x^7 \end{aligned}$$

...and so on

We present and compare exact and numerical results in Table 3.

B. Problem solved using a more exact Lagrange multiplier

As above, we proceed by constructing the Correction functional for (48) as

$$y_{n+1}(x) = y_n(x, t) + \int_0^x \lambda(s)(y_n''''(s) - 4y_n'''(s) - y_n''(s) - 4y_n(s) + 24e^{2s}) ds \quad (52)$$

We take variation of (52) with respect to the independent variable y_n . Using integration by part on the result and making same stationary by setting $\delta y_{n+1}(x) = 0$, we get the following systems.

$$\begin{aligned} \delta y_n: -4\lambda(s) - \lambda''(s) + 4\lambda''''(s) + \lambda''''(s) &= 0 & (a) \\ \delta y_n: \lambda'(s) - 4\lambda''(s) - \lambda''''(s) + 1 &= 0 & (b) \\ \delta y_n': -\lambda(s) + 4\lambda'(s) + \lambda''(s) &= 0 & (c) \\ \delta y_n'': -4\lambda(s) - \lambda'(s) &= 0 & (d) \\ \delta y_n''': \lambda(s) &= 0 & (e) \end{aligned}$$

We proceed by solving the systems (a)-(e) using Operator D method. We use (a) as the differential equation and (b) to (e) as the boundary condition. From (a) we write

$$D^4\lambda + 4D^3\lambda - D^2\lambda - 4\lambda = 0 \tag{53}$$

Which implies that

$$\lambda(s) = A + Be^s + Ce^{-4s} + Fe^{-s} \tag{54}$$

Using (b) – (e) on (54), we determine

$$A = -\frac{1}{4}, \quad B = \frac{1}{10e^x}$$

$$C = -\frac{1}{60e^{-4x}}, \quad F = \frac{1}{6e^{-x}}$$

So that we determine the multiplier as

$$\lambda(s) = -\frac{1}{4} + \frac{1}{10} \frac{e^s}{e^x} - \frac{1}{60} \frac{e^{-4s}}{e^{-4x}} + \frac{1}{6} \frac{e^{-s}}{e^{-x}} \tag{55}$$

And the iteration formula as

$$y_{n+1}(x) = y_n(x, t) + \int_0^x -\frac{1}{4} + \frac{1}{10} \frac{e^s}{e^x} - \frac{1}{60} \frac{e^{-4s}}{e^{-4x}} + \frac{1}{6} \frac{e^{-s}}{e^{-x}} (y_n''''(s) - 4y_n'''(s) - y_n''(s) - 4y_n(s) + 24e2sds) \tag{56}$$

We determine the initial and subsequent approximations as

$$y_0 = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

$$y_1 = \frac{651}{32} + \frac{155}{8}x + \frac{35}{4}x^2 + \frac{7}{3}x^3 - \frac{1}{96}e^{4x} - \frac{64}{3}e^x + \frac{1}{3}x^4 + 2e^{2x}$$

$$y_2 = \frac{49}{4} + \frac{3097}{32}x + \frac{281}{8}x^2 + \frac{91}{12}x^3 + \frac{1}{108}e^{4x} - \frac{3328}{27}e^x + x^4 + \frac{1}{15}x^5 - \frac{1}{288}xe^{4x} + \frac{256}{9}xe^x$$

$$y_3 = \frac{161563}{256} + \frac{56383}{128}x + \frac{9179}{64}x^2 + \frac{677}{24}x^3 - \frac{139}{20736}e^{4x} - \frac{51200}{81}e^x + \frac{43}{12}x^4 + \frac{17}{60}x^5 + \frac{13}{3456}xe^{4x} + \frac{5120}{27}xe^x + \frac{1}{90}x^6 - \frac{1}{1728}x^2e^{4x} + 2e^{2x} - \frac{512}{27}e^xx^2$$

...and so on.

We present and compare exact and numerical results in Table 3 and Fig. 3. We note again from Table 3, the better result using a more exact Lagrange multiplier.

Problem 4:

Consider the following homogeneous nonlinear system [13]

$$u_t + vu_x + u - 1 = 0, \tag{57}$$

$$v_t - uv_x - v - 1 = 0, \tag{58}$$

With initial conditions

$$u(x, 0) = e^x, \quad v(x, 0) = e^{-x}, \tag{59}$$

The exact solutions are

$$u(x, t) = e^{x-t}, \quad v(x, t) = e^{-x+t}, \tag{60}$$

Solution:

A. Problem solved using an approximate Lagrange multiplier

We proceed by constructing two Correction functional of the form:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1(s) \left(\frac{\partial u_n}{\partial s} + v_n \frac{\partial u_n}{\partial x} + \tilde{u}_n - 1 \right) ds \tag{61}$$

$$v_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2(s) \left(\frac{\partial v_n}{\partial s} - u_n \frac{\partial v_n}{\partial x} - \tilde{v}_n - 1 \right) ds \tag{62}$$

But

$$\lambda_1(s) = \lambda_2(s) = -1 \tag{63}$$

So that we get

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left(\frac{\partial u_n}{\partial s} + v_n \frac{\partial u_n}{\partial x} + \tilde{u}_n - 1 \right) ds \tag{64}$$

$$v_{n+1}(x, t) = v_n(x, t) - \int_0^t \left(\frac{\partial v_n}{\partial s} - u_n \frac{\partial v_n}{\partial x} - \tilde{v}_n - 1 \right) ds \tag{65}$$

The following iterations follows:

$$u_0(x, t) = e^x$$

$$v_0(x, t) = e^{-x}$$

$$u_1 := e^x + t - e^{-x} e^x t - e^x t$$

$$v_1 := e^{-x} + t - e^{-x} e^x t + e^{-x} t$$

$$u_2 = e^x + t - e^{-x} e^x t - e^x t - \frac{1}{3} (e^x)^2 e^{-x} t^3 + \frac{1}{3} e^x e^{-x} t^3 + \frac{1}{3} e^x t^3 + \frac{1}{2} t^2 e^{-x} (e^x)^2 + \frac{1}{2} t^2 e^{-x} e^x - \frac{1}{2} t^2$$

$$v_2 = e^{-x} + t - e^{-x} e^x t + e^{-x} t + \frac{1}{3} e^x (e^{-x})^2 t^3 + \frac{1}{3} e^x e^{-x} t^3 - \frac{1}{3} e^{-x} t^3 + \frac{1}{2} e^x (e^{-x})^2 t^2 - \frac{1}{2} t^2 e^{-x} e^x + \frac{1}{2} t^2$$

B. Problem solved using a more exact Lagrange multiplier

Again, we proceed by constructing two Correction functional of the form:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1(s) \left(\frac{\partial u_n}{\partial s} + v_n \frac{\partial u_n}{\partial x} + u_n - 1 \right) ds \quad (66)$$

$$v_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2(s) \left(\frac{\partial v_n}{\partial s} - u_n \frac{\partial v_n}{\partial x} - v_n - 1 \right) ds \quad (67)$$

Using known mathematical manipulations, we get

$$\delta u_{n+1}(x, t) = \delta u_n(x, y, t) + \delta \int_0^t \lambda_1(s) \left(\frac{\partial u_n}{\partial s} + u_n \right) ds \quad (68)$$

$$\delta v_{n+1}(x, t) = \delta v_n(x, y, t) + \delta \int_0^t \lambda_2(s) \left(\frac{\partial v_n}{\partial s} - v_n \right) ds \quad (69)$$

Using integrating by part on the integral sign in (68) & (69) and making the result stationary by setting $\delta u_{n+1}(x, y, t) = \delta v_{n+1}(x, y, t) = 0$ respectively, we get

$$\left. \begin{aligned} -\lambda'_1 + \lambda_1|_{s=x} &= 0 \\ 1 + \lambda_1|_{s=x} &= 0 \end{aligned} \right\} \quad \& \quad \left. \begin{aligned} -\lambda'_2 - \lambda_2|_{s=x} &= 0 \\ 1 + \lambda_2|_{s=x} &= 0 \end{aligned} \right\} \quad (70)$$

From whence we determine the Lagrange multipliers as

$$\lambda_1(s) = -e^{(s-t)}, \lambda_2(s) = -e^{(t-s)}, \quad (71)$$

Substituting (71) into the correction functional, we get the following iteration formulas:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t -e^{(s-t)} \left(\frac{\partial u_n}{\partial s} + v_n \frac{\partial u_n}{\partial x} + u_n - 1 \right) ds \quad (72)$$

$$v_{n+1}(x, t) = v_n(x, t) + \int_0^t -e^{(t-s)} \left(\frac{\partial v_n}{\partial s} - u_n \frac{\partial v_n}{\partial x} - v_n - 1 \right) ds \quad (73)$$

The initial and subsequent iterations are determined as follows.

$$\begin{aligned} u_0(x, t) &= e^x \\ v_0(x, t) &= e^{-x} \\ u_1(x, t) &= e^{x-t} \\ v_1(x, t) &= e^{-x+t} \\ u_2(x, t) &= e^{x-t} \\ v_2(x, t) &= e^{-x+t} \end{aligned}$$

Which is the exact result. We note that only one iteration was needed.

Problem 5:

Consider the following homogeneous nonlinear system [14]

$$u_t + u_x v_x + u_y v_y + u = 0, \quad (74)$$

$$v_t + v_x w_x - v_y w_y - v = 0, \quad (75)$$

$$w_t + w_x u_x + w_y u_y - w = 0, \quad (76)$$

With initial conditions

$$u(x, y, 0) = e^{x+y}, \quad v(x, y, 0) = e^{x-y}, \quad w(x, y, 0) = e^{-x+y} \quad (77)$$

The exact solutions are

$$u(x, y, t) = e^{x+y-t}, \quad v(x, y, t) = e^{x-y+t}, \quad w(x, y, t) = e^{-x+y+t} \quad (78)$$

Solution:

A. Problem solved using an approximate Lagrange multiplier

Wazwaz in [14] had solved the problem using VIM by constructing correction functional of the form:

$$u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^t \lambda_1(s) \left(\frac{\partial u_n}{\partial s} + \frac{\partial u_n}{\partial x} \frac{\partial v_n}{\partial x} + \frac{\partial u_n}{\partial y} \frac{\partial v_n}{\partial y} + \tilde{u}_n \right) ds \quad (79)$$

$$v_{n+1}(x, y, t) = v_n(x, y, t) + \int_0^t \lambda_2(s) \left(\frac{\partial v_n}{\partial s} + \frac{\partial v_n}{\partial x} \frac{\partial w_n}{\partial x} - \frac{\partial v_n}{\partial y} \frac{\partial w_n}{\partial y} - \tilde{v}_n \right) ds \quad (80)$$

$$w_{n+1}(x, y, t) = w_n(x, y, t) + \int_0^t \lambda_3(s) \left(\frac{\partial w_n}{\partial s} + \frac{\partial w_n}{\partial x} \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial y} \frac{\partial u_n}{\partial y} - \tilde{w}_n \right) ds \quad (81)$$

With stationary conditions

$$\left. \begin{matrix} 1 + \lambda_1|_{s=x} = 0 \\ \lambda'_1|_{s=x} = 0 \end{matrix} \right\} \quad \left. \begin{matrix} 1 + \lambda_2|_{s=x} = 0 \\ \lambda'_2|_{s=x} = 0 \end{matrix} \right\} \quad \& \quad \left. \begin{matrix} 1 + \lambda_3|_{s=x} = 0 \\ \lambda'_3|_{s=x} = 0 \end{matrix} \right\} \quad (82)$$

And Lagrange multipliers

$$\lambda_1 = \lambda_2 = \lambda_3 = -1 \quad (83)$$

So that the iteration formulas read

$$u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^t \left(\frac{\partial u_n}{\partial s} + \frac{\partial u_n}{\partial x} \frac{\partial v_n}{\partial x} + \frac{\partial u_n}{\partial y} \frac{\partial v_n}{\partial y} + u_n \right) ds \quad (84)$$

$$v_{n+1}(x, y, t) = v_n(x, y, t) - \int_0^t \left(\frac{\partial v_n}{\partial s} + \frac{\partial v_n}{\partial x} \frac{\partial w_n}{\partial x} - \frac{\partial v_n}{\partial y} \frac{\partial w_n}{\partial y} - v_n \right) ds \quad (85)$$

$$w_{n+1}(x, y, t) = w_n(x, y, t) - \int_0^t \left(\frac{\partial w_n}{\partial s} + \frac{\partial w_n}{\partial x} \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial y} \frac{\partial u_n}{\partial y} - w_n \right) ds \quad (86)$$

Using (77) as the initial approximation and the iteration formulas (84), (85), and (86), the following results are obtained.

$$u_0(x, y, t) = e^{x+y}$$

$$v_0(x, y, t) = e^{x-y}$$

$$w_0(x, y, t) = e^{-x+y}$$

$$u_1(x, y, t) = e^{x+y} - e^{x+y}t$$

$$v_1(x, y, t) = e^{x-y} + e^{x-y}t$$

$$w_1(x, y, t) = e^{-x+y} + e^{-x+y}t$$

$$u_2(x, y, t) = e^{x+y} - e^{x+y}t + \frac{1}{2} e^{x+y}t^2$$

$$v_2(x, y, t) = e^{x-y} + e^{x-y}t + \frac{1}{2} e^{x-y}t^2$$

$$w_2(x, y, t) = e^{-x+y} + e^{-x+y}t + \frac{1}{2} e^{-x+y}t^2$$

$$u_3(x, y, t) = e^{x+y} - e^{x+y}t + \frac{1}{2} e^{x+y}t^2 - \frac{1}{6} e^{x+y}t^3$$

$$v_3(x, y, t) = e^{x-y} + e^{x-y}t + \frac{1}{2} e^{x-y}t^2 + \frac{1}{6} e^{x-y}t^3$$

$$w_3(x, y, t) = e^{-x+y} + e^{-x+y}t + \frac{1}{2} e^{-x+y}t^2 + \frac{1}{6} e^{-x+y}t^3$$

... and so on.

The result clearly tends to the exact solution.

B. Problem solved using a more exact Lagrange multiplier

We now solve the above problem using a more exact Lagrange multiplier by constructing three Correction functional of the form:

$$u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^t \lambda_1(s) \left(\frac{\partial u_n}{\partial s} + \frac{\partial \tilde{u}_n}{\partial x} \frac{\partial \tilde{v}_n}{\partial x} + \frac{\partial \tilde{u}_n}{\partial y} \frac{\partial \tilde{v}_n}{\partial y} + u_n \right) ds \tag{87}$$

$$v_{n+1}(x, y, t) = v_n(x, y, t) + \int_0^t \lambda_2(s) \left(\frac{\partial v_n}{\partial s} + \frac{\partial \tilde{v}_n}{\partial x} \frac{\partial \tilde{w}_n}{\partial x} - \frac{\partial \tilde{v}_n}{\partial y} \frac{\partial \tilde{w}_n}{\partial y} - v_n \right) ds \tag{88}$$

$$w_{n+1}(x, y, t) = w_n(x, y, t) + \int_0^t \lambda_3(s) \left(\frac{\partial w_n}{\partial s} + \frac{\partial \tilde{w}_n}{\partial x} \frac{\partial \tilde{u}_n}{\partial x} + \frac{\partial \tilde{w}_n}{\partial y} \frac{\partial \tilde{u}_n}{\partial y} - w_n \right) ds \tag{89}$$

To determine the Lagrange multipliers we take variation of (87), (88) & (89) w.r.t the independent variables u_n , v_n & w_n respectively nothing that \tilde{u}_n , \tilde{v}_n & \tilde{w}_n are restricted variation, that is $\delta \tilde{u}_n = 0$, $\delta \tilde{v}_n = 0$ & $\delta \tilde{w}_n = 0$

So that we get

$$\delta u_{n+1}(x, y, t) = \delta u_n(x, y, t) + \delta \int_0^t \lambda_1(s) \left(\frac{\partial u_n}{\partial s} + u_n \right) ds \tag{90}$$

$$\delta v_{n+1}(x, y, t) = \delta v_n(x, y, t) + \delta \int_0^t \lambda_2(s) \left(\frac{\partial v_n}{\partial s} - v_n \right) ds \tag{91}$$

$$\delta w_{n+1}(x, y, t) = \delta w_n(x, y, t) + \delta \int_0^t \lambda_3(s) \left(\frac{\partial w_n}{\partial s} - w_n \right) ds \tag{92}$$

Using integration by part and making the result stationary by setting

$$\delta u_{n+1}(x, y, t) = \delta v_{n+1}(x, y, t) = \delta w_{n+1}(x, y, t) = 0 \text{ we get}$$

$$\left. \begin{matrix} -\lambda'_1 + \lambda_1|_{s=x} = 0 \\ 1 + \lambda_1|_{s=x} = 0 \end{matrix} \right\} \quad \left. \begin{matrix} -\lambda'_2 - \lambda_2|_{s=x} = 0 \\ 1 + \lambda_2|_{s=x} = 0 \end{matrix} \right\} \quad \& \quad \left. \begin{matrix} -\lambda'_3 - \lambda_3|_{s=x} = 0 \\ 1 + \lambda'_3|_{s=x} = 0 \end{matrix} \right\} \tag{93}$$

Solving (93) using Integrating factor, the exact Lagrange multipliers are determined as

$$\lambda_1(s) = -e^{(s-t)}, \lambda_2(s) = -e^{(t-s)} \text{ \& } \lambda_3(s) = -e^{(t-s)} \tag{94}$$

Substituting the Lagrange multipliers into (87) - (89), we get the following iteration formulas

$$u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^t (e^{(s-t)}) \left(\frac{\partial u_n}{\partial s} + \frac{\partial u_n}{\partial x} \frac{\partial v_n}{\partial x} + \frac{\partial u_n}{\partial y} \frac{\partial v_n}{\partial y} + u_n \right) ds \tag{95}$$

$$v_{n+1}(x, y, t) = v_n(x, y, t) - \int_0^t (e^{(t-s)}) \left(\frac{\partial v_n}{\partial s} + \frac{\partial v_n}{\partial x} \frac{\partial w_n}{\partial x} - \frac{\partial v_n}{\partial y} \frac{\partial w_n}{\partial y} - v_n \right) ds \tag{96}$$

$$w_{n+1}(x, y, t) = w_n(x, y, t) - \int_0^t (e^{(t-s)}) \left(\frac{\partial w_n}{\partial s} + \frac{\partial w_n}{\partial x} \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial y} \frac{\partial u_n}{\partial y} - w_n \right) ds \tag{97}$$

Using (77) as the initial approximation and the iteration formulas (95), (96), and (97), we obtain the following results.

$$\begin{aligned} u_0(x, y, t) &= e^{x+y} \\ v_0(x, y, t) &= e^{x-y} \\ w_0(x, y, t) &= e^{-x+y} \\ u_1(x, y, t) &= e^{x+y-t} \\ v_1(x, y, t) &= e^{x-y+t} \\ w_1(x, y, t) &= e^{-x+y+t} \\ u_2(x, y, t) &= e^{x+y-t} \\ v_2(x, y, t) &= e^{x-y+t} \\ w_2(x, y, t) &= e^{-x+y+t} \end{aligned}$$

We observe that by using the exact Lagrange multiplier for the problem, the result converged to the exact solution after a single iteration. A major significant difference between Wazwaz solution and ours is found in the choice of the correction functional we used and the fact that we only applied restricted variation to the non-linear terms in the correction functional in the determination of a more exact Lagrange multiplier.

This was also the case for all other problems considered. Restricted variation was only applied to the nonlinear term(s) in the accurate determination of the Lagrange multiplier.

Table 1. Comparison of exact solution and VIM solution for problem 1

<i>x</i>	<i>Exact</i>	<i>VIM 6th Iteration (with approx. λ(s))</i>	<i>Absolute Error</i>
0.00	0.0000000	0.0000000	0.0000e+00
0.10	0.1398334	0.1398334	0.0000e+00
0.20	0.3586693	0.3586693	0.0000e+00
0.30	0.6555202	0.6555202	0.0000e+00
0.40	1.0294183	1.0294183	0.0000e+00
0.50	1.4794255	1.4794255	0.0000e+00
0.60	2.0046425	2.0046425	0.0000e+00
0.70	2.6042177	2.6042177	0.0000e+00
0.80	3.2773561	3.2773561	0.0000e+00
0.90	4.0233269	4.0233269	0.0000e+00
1.00	4.8414710	4.8414710	0.0000e+00

Table 2. Comparison of exact solution and VIM solutions for problem 2

<i>x</i>	<i>Exact</i>	<i>VIM 14th Iteration (With approx. λ(s))</i>	<i>VIM 1st Iteration (With exact λ(s))</i>	<i>Absolute Error (With approx. λ(s))</i>	<i>Absolute Error (With exact λ(s))</i>
0.00	7.0000000	7.0000000	7.0000000	0.0000e+00	0.0000e+00
0.10	6.3519587	6.3519587	6.3519587	1.0000e-09	0.0000e+00
0.20	5.7966137	5.7966137	5.7966137	0.0000e+00	2.0000e-09
0.30	5.3190748	5.3190748	5.3190748	1.0000e-09	3.0000e-09
0.40	4.9067427	4.9067427	4.9067427	0.0000e+00	1.0000e-09
0.50	4.5489799	4.5489799	4.5489799	0.0000e+00	0.0000e+00
0.60	4.2368258	4.2368258	4.2368258	0.0000e+00	0.0000e+00
0.70	3.9627507	3.9627507	3.9627507	0.0000e+00	2.0000e-09
0.80	3.7204438	3.7204438	3.7204438	1.0000e-09	1.0000e-09
0.90	3.5046305	3.5046305	3.5046305	1.2000e-08	1.0000e-09
1.00	3.3109150	3.3109149	3.3109150	8.5000e-08	0.0000e+00

Table 2 compares the result of the exact and numerical solutions for problem 2, obtained using VIM. We note from the table, that the numerical result obtained using an approximate Lagrange multiplier for VIM are good, but fourteen (14) iterations were needed as against one (1) iteration required when using a more accurate Lagrange multiplier. It is also observed that the result of the more accurate Lagrange multiplier is a better approximate for the given problem.

Table 3. Comparison of exact solution and VIM solutions for problem 3

x	<i>Exact</i>	<i>VIM 3rd Iteration (With less exact $\lambda(s)$)</i>	<i>VIM 3rd Iteration (With exact $\lambda(s)$)</i>	<i>Absolute Error (With less exact $\lambda(s)$)</i>	<i>Absolute Error (With exact $\lambda(s)$)</i>
0.00	1.0000000	1.0000000	1.0000000	0.0000e+00	0.0000e+00
0.10	1.2214028	1.2214027	1.2214029	1.8000e-08	1.5800e-07
0.20	1.4918247	1.4918219	1.4918248	2.7980e-06	9.8000e-08
0.30	1.8221188	1.8220694	1.8221186	4.9350e-05	2.0000e-07
0.40	2.2255409	2.2251576	2.2255407	3.8331e-04	2.7200e-07
0.50	2.7182818	2.7163860	2.7182816	1.8958e-03	2.7200e-07
0.60	3.3201169	3.3130668	3.3201170	7.0501e-03	1.2300e-07
0.70	4.0552000	4.0336617	4.0551999	2.1538e-02	3.3000e-08
0.80	4.9530324	4.8960391	4.9530318	5.6993e-02	5.7600e-07
0.90	6.0496475	5.9144851	6.0496469	1.3516e-01	5.3400e-07
1.00	7.3890561	7.0949995	7.3890552	2.9406e-01	8.9900e-07

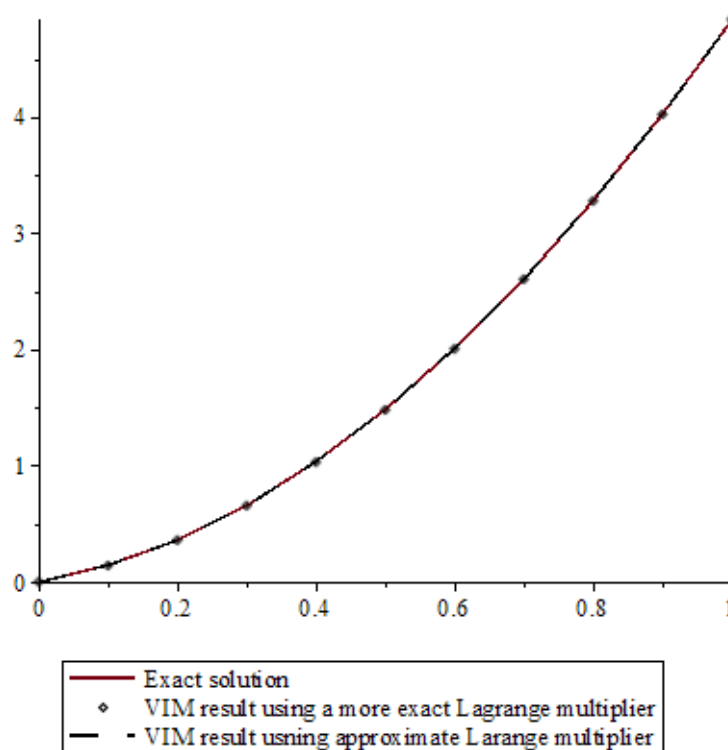


Fig. 1. Graph comparing results of the exact solution, the solution of the 1st iteration of VIM using a more exact Lagrange multiplier, and the solution of the 6th iteration of VIM using an approximate Lagrange multiplier for Problem 1

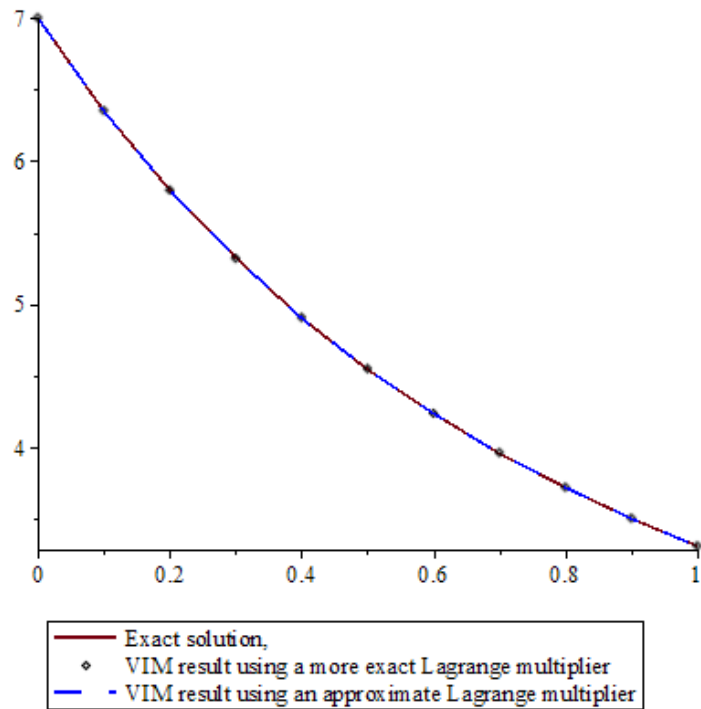


Fig. 2. Graph comparing results of the exact solution, and the solution of the 1st iteration of VIM using a more exact Lagrange multiplier and the solution of the 14th iteration of VIM using an approximate Lagrange multiplier for problem 2

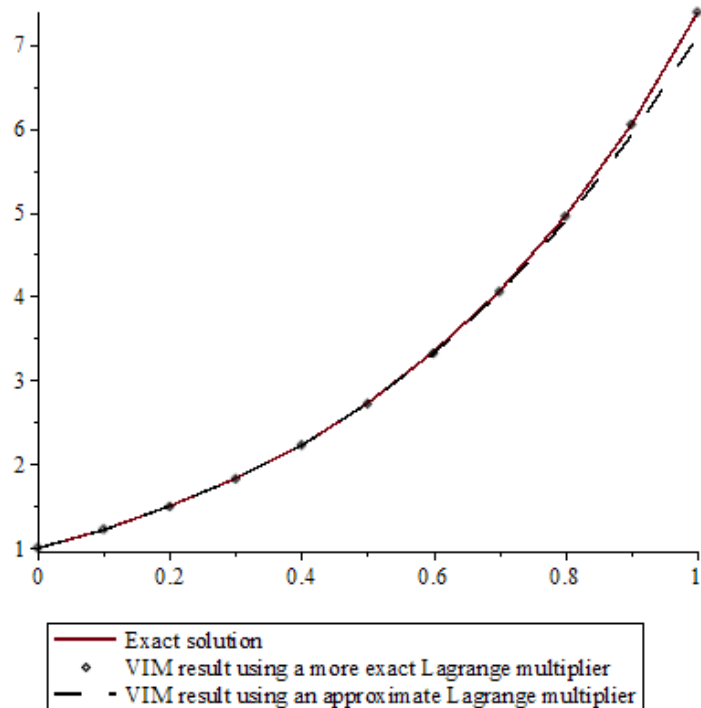


Fig. 3. Graph comparing results of the exact solution, and the solution of the 3rd iteration of VIM using a more exact Lagrange multiplier and the solution of the 3rd iteration of VIM using an approximate Lagrange multiplier for problem 3

4 Conclusion

Owing to its relative ease of use, the formula (1) has gained popularity in recent times among authors while employing VIM. For some class of problems however, there is need to resort to other ways of determining more accurate Lagrange multiplier for VIM as the formula (1) may not always do so. With the foregoing serving as motivation, this paper has focused on the computation of accurate Lagrange multiplier for VIM. It has demonstrated that operator D-method and integrating factor can be used in solving the resulting systems after applying variational theories to the correction functional when using VIM. The obtained result shows clearly how the accurate determination of the Lagrange multiplier can significantly reduce the number of iterations required to get good approximate results or convergence to the exact result with few iterations.

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Competing Interests

Authors have declared that no competing interests exist.

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