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A Class of Explicit Integrators with off-grid Interpolation for Solving Non-linear Systems of First Order ODEs

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This research work is aimed at constructing a class of explicit integrators with improved stability and accuracy by incorporating an off-gird interpolation point for the purpose of making them efficient for solving stiff initial value problems. Accordingly, continuous formulations of a class of hybrid explicit integrators are derived using multi-step collocation method through matrix inversion technique, for step numbers $k = 2, 3, 4$. The discrete schemes were deduced from their respective continuous formulations. The stability and convergence analysis were carried out and shown to be A(α)-stable and convergent respectively. The discrete schemes when implemented as block integrators to solve some non-linear problems, it was observed that the results obtained compete favorably with the MATLAB ode23 solver.

Keywords: Block; Hybrid; Explicit integrators; off-gird interpolation; continuous formulation.

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1 Introduction

Most of the ordinary differential equations (ODEs) arising from modeling real life problems cannot be solved analytically, hence the need to seek for approximate solutions using numerical methods. As such, the importance of numerical methods cannot be over emphasized. To solve stiff ODEs, various scholars have made several attempts to come up with various methods of solution. Consider the initial value problem (IVP)

$$y' = f(x, y), \;\; y(x_0) = y_0 \;\; a \leq x \leq b \tag{1.1}$$

where a and b are finite, with the assumption that (1.1) has a unique continuously differentiable solution $y(x)$. So far common numerical methods that have been developed for solving (1.1) are one-step methods. On the other hand, multistep methods attempt to gain efficiency by keeping and using the information from the previous steps. A method is called Linear Multi-step Method (LMM) if a linear combination of the values of the computed solution and possibly its derivative in the previous points are used. A k - step LMM is given as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}) \tag{1.2}$$

where α_j, β_j are constants called the coefficients of the method with the assumption that $\alpha_0^2 + \beta_0^2 > 0$ and $\alpha_k \neq 0$. The method is explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$. Most conventional methods have self starting value issues which could lead to growing numerical errors and corrupting further approximations [1]. To resolve this issue, [2] proposed block Linear Multistep methods based on the multi-step collocation approach of Lie and Norsett [3]. These methods were developed through the continuous formulation of the linear k-step methods which provided sufficient number of simultaneous discrete methods to be used as single integrators. These methods have been useful in handling stiff equations due to their better stability properties. The following researchers, [4, 5, 6, 7] and [3] have developed block linear multi-step methods that have better stability properties. Implicit LMMs like backward differentiation formulae (BDF) have been considered to be best for solution of stiff initial value problems. However, the explicit linear multi-step methods enjoy some advantages comparing to the implicit methods. The most important advantage of an explicit method is that there is no need to solve any implicit system or involve any iterative procedure in each time step [8]. Thus, it involves far less computational effort in each time step when compared to an implicit method [9, 10, 11]. Researchers like [12, 13, 14, 15] among others have developed reliable explict methods for stiff ODEs. Explicit methods in general, are considered to be inefficient for solving stiff problems, due to their low accuracy and poor stability properties. Consequently, this paper is aimed at constructing explicit methods with improved accuracy and better stability properties, by incorporating an off-grid interpolation point. The first section has the introduction, the second includes the derivation techniques. In the third section, the convergence and stability analysis are carried out while in the last section we test the strength of these new methods by solving some non-linear ODEs.

2 Derivation Techniques

2.1 Derivation of multistep collocation method

The method carried out by Onumayi et al. [2] shall be used in this derivation, where a k-step collocation method was obtained as:

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \beta_j(x) f(x_j, y(x_j)), \quad x_n \leq x \leq x_{n+k} \tag{2.1}$$

where t denotes the number of interpolation points and m denotes the number of distinct collocation points. The continuous coefficients of (2.1), $\alpha_j(x)$ and $\beta_j(x)$ are defined as;

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^i, \quad j \in \{0, 1, \cdots, t-1\} \tag{2.2}$$

$$h\beta_j(x) = h \sum_{i=0}^{t+m-1} \beta_{j,i+1} x^i, \quad j \in \{0, 1, \cdots, m-1\} \tag{2.3}$$

To get $\alpha_j(x)$ and $\beta_j(x)$, [2], arrived at a matrix equation of the form:

$$DC = I \tag{2.4}$$

where I is the identity matrix of dimension $(t+m) \times (t+m)$ while D and C are matrices defined as;

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_0 & \cdots & (t+m-1)x_0^{t+m-2} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 1 & 2x_{m-1} & \cdots & (t+m-1)x_{m-1}^{t+m-2} \end{bmatrix} \tag{2.5}$$

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & \cdots & h\beta_{m-1,t+m} \end{bmatrix} \tag{2.6}$$

From (2.4) it follows that $C = D^{-1}$, where the columns of C gives the continuous coefficients of the continuous scheme (2.1). Using this idea, the continuous formulation of the explicit method with an off-grid interpolation point is given as;

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + \alpha_\mu(x) y_{n+\mu} + h[\beta_{k-1}(x) f(x_{n+k-1}, y(x_{n+k-1}))] \tag{2.7}$$

where $\mu \notin \{0, k\}$

2.2 Derivation of continuous formulation for two-step explicit method incorporating one off-Grid interpolation point

In this method, we incorporate one off-grid point at $x = x_{n+\frac{7}{4}}$ as interpolation point, thus $k = 2, t = 3, m = 1$ and (2.7) becomes

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_{\frac{7}{4}}(x)y_{n+\frac{7}{4}} + h\beta_1(x)f_{n+1} \tag{2.8}$$

and the D matrix in (2.5) becomes;

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 \\ 1 & x_n + \frac{7}{4}h & (x_n + \frac{7}{4}h)^2 & (x_n + \frac{7}{4}h)^3 \\ 0 & 1 & 2x_n + 2h & 3(x_n + h)^2 \end{bmatrix} \tag{2.9}$$

Using maple 18 software, the inverse $C = D^{-1}$ of the D matrix is obtained, which gives the continuous scheme as;

$$\begin{gathered} y(x) = -\tfrac{6}{7}\tfrac{1}{h^3}[3h^2 - 5h(x - x_n) + 2(x - x_n)^2]y_n + \tfrac{2}{9}\tfrac{1}{h^3}[7h^2 - h(x - x_n) - 6(x - xn)^2]y_{n+1} + \\ \tfrac{64}{63}\tfrac{1}{h^3}[h^2 - 4h(x - x_n) + 3(x - x_n)^2]y_{n+\frac{7}{4}} - \tfrac{1}{3}\tfrac{1}{h^2}[7h^2 - 22h(x - x_n) + 12(x - x_n)^2]f_{n+1} \end{gathered} \tag{2.10}$$

Evaluating (2.10) at $x = x_{n+2}$, and its derivative at $x = x_{n+\frac{7}{4}}, x_{n+2}$ we obtain;

$$\begin{aligned} y_{n+2} &= \frac{1}{7}y_n - \frac{8}{9}y_{n+1} + \frac{128}{63}y_{n+\frac{7}{4}} - \frac{2}{3}hf_{n+1} \\ y_{n+\frac{7}{4}} &= \frac{27}{272}y_n + \frac{245}{272}y_{n+1} + \frac{147}{272}hf_{n+1} + \frac{21}{68}hf_{n+\frac{7}{4}} \\ y_{n+1} &= -\frac{27}{133}y_n + \frac{160}{133}y_{n+\frac{7}{4}} - \frac{33}{38}hf_{n+1} - \frac{9}{38}hf_{n+2} \end{aligned} \tag{2.11}$$

2.3 Derivation of continuous formulation for three-step explicit method incorporating one off-grid interpolation point

In this method, we incorporate one off-grid point at $x = x_{n+\frac{11}{4}}$ as interpolation point, thus $k = 3, t = 4, m = 1$ and (2.7) becomes;

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_{\frac{11}{4}}(x)y_{n+\frac{11}{4}} + h\beta_2(x)f_{n+2} \tag{2.12}$$

and the D matrix in (2.5) becomes;

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 \\ 1 & x_n + 2h & (x_n + 2h)^2 & (x_n + 2h)^3 & (x_n + 2h)^4 \\ 1 & x_n + \frac{11}{4}h & (x_n + \frac{11}{4}h)^2 & (x_n + \frac{11}{4}h)^3 & (x_n + \frac{11}{4}h)^4 \\ 0 & 1 & 2x_n + 4h & 3(x_n + 2h)^2 & 4(x_n + 2h)^3 \end{bmatrix} \tag{2.13}$$

Using Maple software, the inverse $C = D^{-1}$ of the D matrix is obtained, which gives the continuous scheme as;

$$\begin{gathered} y(x) = -\tfrac{1}{44}\tfrac{1}{h^4}[104h^3 - 174h^2(x - x_n) + 93h(x - x_n)^2 - 16(x - x_n)^3]y_n + \\ \tfrac{1}{7}\tfrac{1}{h^4}[44h^3 - 120h^2(x - x_n) + 81h(x - x_n)^2 - 16(x - x_n)^3]y_{n+1} - \\ \tfrac{1}{36}\tfrac{1}{h^4}[88h^3 - 262h^2(x - x_n) + 141h(x - x_n)^2 - 16(x - x_n)^3]y_{n+2} - \\ \tfrac{256}{593}\tfrac{1}{h^4}[4h^3 - 16h^2(x - x_n) + 15h(x - x_n)^2 - 4(x - x_n)^3]y_{n+\frac{11}{4}} + \\ \tfrac{1}{6}\tfrac{1}{h^3}[22h^3 - 82h6^2(x - x_n) + 69h(x - x_n)^2 - 16(x - x_n)^3]f_{n+2} \end{gathered} \tag{2.14}$$

Evaluating (2.14) at $x = x_{n+3}$, and its derivative at $x = x_{n+1}, x_{n+\frac{11}{4}}, x_{n+3}$ we obtain;

$$\begin{aligned}
y_{n+3} &= \frac{1}{22}y_n - \frac{3}{7}y_{n+1} - \frac{5}{6}y_{n+2} + \frac{512}{231}y_{n+\frac{11}{4}} - hf_{n+2} \\
y_{n+\frac{11}{4}} &= -\frac{1323}{53248}y_n + \frac{3267}{13312}y_{n+1} + \frac{41503}{53248}y_{n+2} + \frac{17787}{26624}hf_{n+2} + \frac{231}{832}hf_{n+\frac{11}{4}} \\
y_{n+2} &= \frac{117}{1539}y_n - \frac{684}{973}y_{n+1} + \frac{17408}{10703}y_{n+\frac{11}{4}} - \frac{210}{139}hf_{n+2} - \frac{36}{139}hf_{n+3} \\
y_{n+1} &= -\frac{49}{484}y_n + \frac{343}{396}y_{n+2} + \frac{256}{1089}y_{n+\frac{11}{4}} - \frac{7}{11}hf_{n+1} - \frac{49}{66}hf_{n+2}
\end{aligned} \tag{2.15}$$

2.4 Derivation of continuous formulation for four-step explicit method incorporating one off-grid interpolation point

In this method, we incorporate one off-grid point at $x = x_{n+\frac{15}{4}}$ as interpolation point, thus $k = 4, t = 5, m = 1$ and (2.7) becomes;

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_3(x)y_{n+3} + \alpha_{\frac{15}{4}}(x)y_{n+\frac{15}{4}} + h\beta_3(x)f_{n+3} \tag{2.16}$$

and the D matrix in (2.5) becomes;

$$D = \begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\
1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 & (x_n + h)^5 \\
1 & x_n + 2h & (x_n + 2h)^2 & (x_n + 2h)^3 & (x_n + 2h)^4 & (x_n + 2h)^5 \\
1 & x_n + 3h & (x_n + 3h)^2 & (x_n + 3h)^3 & (x_n + 3h)^4 & (x_n + 3h)^5 \\
1 & x_n + \frac{15}{4}h & (x_n + \frac{15}{4}h)^2 & (x_n + \frac{15}{4}h)^3 & (x_n + \frac{15}{4}h)^4 & (x_n + \frac{15}{4}h)^5 \\
0 & 1 & 2x_n + 6h & 3(x_n + 3h)^2 & 4(x_n + 3h)^3 & 5(x_n + 3h)^4
\end{bmatrix} \tag{2.17}$$

Using Maple software, the inverse $C = D^{-1}$ of the D matrix is obtained, which gives the continuous scheme as;

$$\begin{aligned}
y(x) = {} & \tfrac{1}{270}\tfrac{1}{h^5}[270h^5 - 657h^4(x - x_n) + 591h^3(x - x_n)^2 - 251h^2(x - x_n)^3 + 51h(x - x_n)^4 - 4(x - x_n)^5]y_n \\
& + \tfrac{1}{44}\tfrac{1}{h^5}[270h^4(x - x_n) - 387h^3(x - x_n)^2 + 204h^3(x - x_n)^3 - 47h(x - x_n)^4 + 4(x - x_n)^5]y_{n+1} \\
& - \tfrac{1}{14}\tfrac{1}{h^5}[135h^4(x - x_n) - 261h^3(x - x_n)^2 + 165h^2(x - x_n)^3 - 43h(x - x_n)^4 + 4(x - x_n)^5]y_{n+2} \\
& + \tfrac{1}{36}\tfrac{1}{h^5}[150h^4(x - x_n) - 295h^3(x - x_n)^2 + 188h^2(x - x_n)^3 - 47h(x - x_n)^4 + 4(x - x_n)^5]y_{n+3} \\
& + \tfrac{1024}{10395}\tfrac{1}{h^5}[18h^4(x - x_n) - 39h^3(x - x_n)^2 + 29h^2(x - x_n)^3 - 9h(x - x_n)^4 + (x - x_n)^5]y_{n+\frac{15}{4}} \\
& - \tfrac{1}{18}\tfrac{1}{h^4}[90h^4(x - x_n) - 189h^3(x - x_n)^2 + 134h^2(x - x_n)^3 - 39h(x - x_n)^4 + 4(x - x_n)^5]f_{n+3}
\end{aligned} \tag{2.18}$$

Evaluating (2.18) at $x = x_{n+4}$, and its derivative at $x = x_{n+1}, x_{n+2}, x_{n+\frac{15}{4}}, x_{n+4}$ we obtain;

$$\begin{aligned}
y_{n+4} &= -\frac{1}{45}y_n + \frac{2}{11}y_{n+1} - \frac{6}{7}y_{n+2} - \frac{2}{3}y_{n+3} + \frac{8192}{3465}y_{n+\frac{15}{4}} - \frac{4}{3}hf_{n+3} \\
y_{n+\frac{15}{4}} &= \frac{5929}{571904}y_n - \frac{99225}{1143808}y_{n+1} + \frac{245025}{571904}y_{n+2} + \frac{741125}{1143808}y_{n+3} + \frac{444675}{571904}hf_{n+3} + \frac{1155}{4468}hf_{n+\frac{15}{4}} \\
y_{n+3} &= -\frac{41}{735}y_n + \frac{243}{539}y_{n+1} - \frac{711}{343}y_{n+2} + \frac{151552}{56595}y_{n+\frac{15}{4}} - \frac{146}{49}hf_{n+3} - \frac{18}{49}hf_{n+4} \\
y_{n+2} &= \frac{49}{2025}y_n - \frac{49}{165}y_{n+1} + \frac{49}{45}y_{n+3} + \frac{4096}{22275}y_{n+\frac{15}{4}} - \frac{14}{15}hf_{n+2} - \frac{98}{138}hf_{n+3} \\
y_{n+1} &= -\frac{242}{2025}y_n + \frac{242}{105}y_{n+2} - \frac{121}{135} - \frac{4096}{14175}y_{n+\frac{15}{4}} - \frac{11}{15}hf_{n+1} + \frac{121}{135}hf_{n+3}
\end{aligned} \tag{2.19}$$

3 Analysis of the New Block Methods

Here convergence analysis and plots of region of absolute stability of the newly constructed methods are considered.

3.1 Zero stability analysis of the new block explicit methods

Following Fatunla [16], the block schemes can be represented as:

$$A^{(1)}y_{n+i} = A^{(0)}y_{n+i} + hB^{(1)}f_{n+i} \tag{3.1}$$

The block method (2.11), expressed as (3.1):

where;

$$A^{(1)} = \begin{bmatrix} 1 & -\frac{160}{133} & 0 \\ -\frac{245}{275} & 1 & 0 \\ \frac{8}{9} & \frac{128}{63} & 1 \end{bmatrix}, A^{(0)} = \begin{bmatrix} 0 & 0 & \frac{27}{133} \\ 0 & 0 & -\frac{27}{272} \\ 0 & 0 & \frac{1}{7} \end{bmatrix}, B^{(1)} = \begin{bmatrix} -\frac{33}{38} & 0 & -\frac{9}{38} \\ \frac{147}{272} & \frac{21}{68} & 0 \\ -\frac{2}{3} & 0 & 0 \end{bmatrix}$$

The first characteristic polynomial of the block method (2.11) is given by

$$\rho(\lambda) = det(\lambda A^{(1)} - A^{(0)})$$

$$\Rightarrow |\lambda A^{(1)} - A^{(0)}| = 0$$

$$\Rightarrow \rho(\lambda) = \left| \lambda \begin{bmatrix} 1 & -\frac{160}{133} & 0 \\ -\frac{245}{272} & 1 & 0 \\ \frac{8}{9} & -\frac{128}{63} & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \frac{27}{133} \\ 0 & 0 & -\frac{27}{272} \\ 0 & 0 & \frac{1}{7} \end{bmatrix} \right|$$

$$\Rightarrow -\frac{27}{323}\lambda^3 - \frac{27}{323}\lambda^2 = 0$$

$$\therefore \lambda = -1, 0, 0$$

Since $|\lambda_i| \leq 1, i = 1, 2, 3$ then, by Fatunla [16], the block method (2.11) is zero stable.

Similarly the block explicit method (2.15) given as (3.1):

where;

$$A^{(1)} = \begin{bmatrix} 1 & -\frac{343}{396} & -\frac{256}{1089} & 0 \\ \frac{684}{973} & 1 & -\frac{17408}{10703} & 0 \\ -\frac{3267}{13312} & -\frac{41503}{53248} & 1 & 0 \\ \frac{3}{7} & \frac{5}{6} & -\frac{512}{231} & 1 \end{bmatrix}, A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{49}{484} \\ 0 & 0 & 0 & -\frac{117}{1529} \\ 0 & 0 & 0 & \frac{1323}{53248} \\ 0 & 0 & 0 & -\frac{1}{22} \end{bmatrix}, B^{(1)} = \begin{bmatrix} -\frac{7}{11} & -\frac{49}{66} & 0 & 0 \\ 0 & -\frac{210}{139} & 0 & -\frac{36}{139} \\ 0 & \frac{17789}{26624} & \frac{231}{832} & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

The first characteristic polynomial of the block method (2.15) is given by

$$\rho(\lambda) = det(\lambda A^{(1)} - A^{(0)})$$

$$\Rightarrow |\lambda A^{(1)} - A^{(0)}| = 0$$

$$\Rightarrow \rho(\lambda) = \left| \lambda \begin{bmatrix} 1 & -\frac{343}{396} & -\frac{256}{1089} & 0 \\ \frac{684}{973} & 1 & -\frac{17408}{10703} & 0 \\ -\frac{3267}{13312} & -\frac{41503}{53248} & 1 & 0 \\ \frac{3}{7} & \frac{5}{6} & -\frac{512}{231} & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \frac{49}{484} \\ 0 & 0 & 0 & -\frac{117}{1529} \\ 0 & 0 & 0 & \frac{1323}{53248} \\ 0 & 0 & 0 & -\frac{1}{22} \end{bmatrix} \right|$$

$$\Rightarrow \frac{1323}{19877}\lambda^4 + \frac{1323}{19877}\lambda^3 = 0$$

$$\therefore \lambda = -1, 0, 0, 0$$

Since $|\lambda_i| \leq 1, i = 1, 2, 3, 4$ then, by Fatunla [16], the block method (2.15) is zero stable.

Similarly the block explicit method (2.19) expressed as (3.1):

where;

$$A^{(1)} = \begin{bmatrix} 1 & -\frac{242}{105} & \frac{121}{135} & \frac{4096}{14175} & 0 \\ \frac{49}{165} & 1 & -\frac{49}{45} & -\frac{4096}{22275} & 0 \\ -\frac{243}{539} & \frac{711}{343} & 1 & -\frac{151552}{56595} & 0 \\ \frac{99225}{1143808} & -\frac{245025}{571904} & -\frac{741125}{1143808} & 1 & 0 \\ -\frac{2}{11} & \frac{6}{7} & \frac{2}{3} & -\frac{8192}{3465} & 1 \end{bmatrix}, A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{242}{2025} \\ 0 & 0 & 0 & 0 & -\frac{49}{2025} \\ 0 & 0 & 0 & 0 & \frac{41}{735} \\ 0 & 0 & 0 & 0 & -\frac{5929}{571904} \\ 0 & 0 & 0 & 0 & \frac{1}{45} \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} -\frac{11}{15} & 0 & \frac{121}{135} & 0 & 0 \\ 0 & -\frac{14}{15} & -\frac{98}{135} & 0 & 0 \\ 0 & 0 & -\frac{146}{49} & 0 & -\frac{18}{49} \\ 0 & 0 & \frac{444675}{571904} & \frac{1155}{4468} & 0 \\ 0 & 0 & -\frac{4}{3} & 0 & 0 \end{bmatrix}$$

The first characteristic polynomial of the block method (2.19) is given by

$$\rho(\lambda) = det(\lambda A^{(1)} - A^{(0)})$$

$$\Rightarrow |\lambda A^{(1)} - A^{(0)}| = 0$$

$$\Rightarrow \rho(\lambda) = \left| \lambda \begin{bmatrix} 1 & -\frac{242}{105} & \frac{121}{135} & \frac{4096}{14175} & 0 \\ \frac{49}{165} & 1 & -\frac{49}{45} & -\frac{4096}{22275} & 0 \\ -\frac{243}{539} & \frac{711}{343} & 1 & -\frac{151552}{56595} & 0 \\ \frac{99225}{1143808} & -\frac{245025}{571904} & -\frac{741125}{1143808} & 1 & 0 \\ -\frac{2}{11} & \frac{6}{7} & \frac{2}{3} & -\frac{8192}{3465} & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{242}{2025} \\ 0 & 0 & 0 & 0 & -\frac{49}{2025} \\ 0 & 0 & 0 & 0 & \frac{41}{735} \\ 0 & 0 & 0 & 0 & -\frac{5929}{571904} \\ 0 & 0 & 0 & 0 & \frac{1}{45} \end{bmatrix} \right|$$

$$\Rightarrow -\frac{583443000}{627952168909}\lambda^5 - \frac{583443000}{627952168909}\lambda^4 = 0$$

$$\therefore \lambda = -1, 0, 0, 0, 0$$

Since $|\lambda_i| \leq 1, i = 1, 2, 3, 4, 5$ then, by Fatunla [16], the block method (2.19) is zero stable.

3.2 Order and error constant of the new block Hybrid explicit methods, k=2,3,4.

Table 1. Order and Error Constant of Scheme (2.11), (2.15), and (2.19)

Evaluating points			Order			Error Constants		
k=2	**k=3**	**k=4**	**k=2**	**k=3**	**k=4**	**k=2**	**k=3**	**k=4**
$y'(x = x_{n+2})$	$y'(x = x_{n+1})$	$y'(x = x_{n+1})$	**3**	**4**	**5**	$\frac{39}{1216}$	$-\frac{49}{5280}$	$\frac{121}{10800}$
$y'(x = x_{n+\frac{7}{4}})$	$y'(x = x_{n+3})$	$y'(x = x_{n+2})$	**3**	**4**	**5**	$-\frac{441}{34816}$	$\frac{123}{5560}$	$-\frac{121}{10800}$
$y(x = x_{n+2})$	$y'(x = x_{n+\frac{11}{4}})$	$y'(x = x_{n+4})$	**3**	**4**	**5**	$\frac{1}{48}$	$-\frac{53361}{8519680}$	$\frac{17}{784}$
	$y(x = x_{n+3})$	$y'(x = x_{n+\frac{15}{4}})$		**4**	**5**		$\frac{1}{80}$	$-\frac{266805}{73203712}$
		$y(x = x_{n+4})$			**5**			$\frac{1}{120}$

By the analysis above, the block method for $k = 2, 3, 4$, are zero stable and has order $p > 1$. Thus by Henrici [17], the block explicit methods (2.11), (2.15), and (2.19) are convergent.

3.3 Plots of stability region of the new methods

The stability analysis and plots of region of absolute stability for schemes (2.11), (2.15) and (2.19) will be considered. Using Maple software, we obtain matrix $P = r * (A^{(1)} - z * B^{(1)}) - A^{(0)}$, where $A^{(1)}, B^{(1)}$, and $A^{(0)}$ are as defined in the previous section. The determinant of P was obtained and the derivative of the determinant was computed for the schemes. This information was used in a MATLAB code to obtain the region of absolute stability for each of the methods. The region of absolute stability is the area outside the enclosure in Fig. 1, 2 and 3 below.

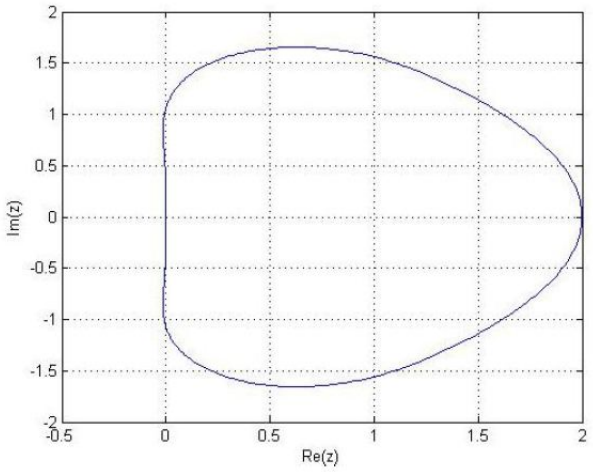

Fig. 1. The region of absolute stability for Scheme (2.11)

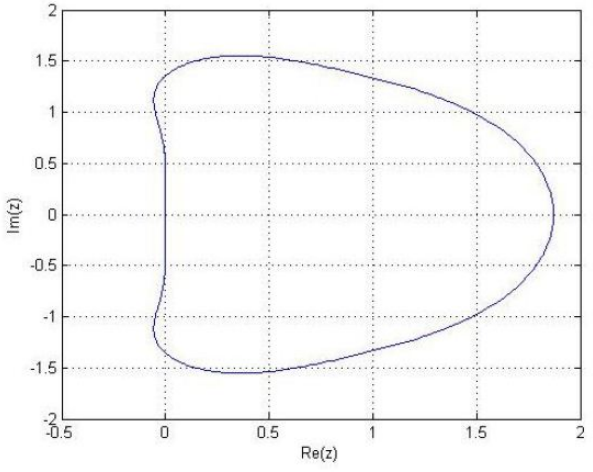

Fig. 2. The region of absolute stability for Scheme (2.15)

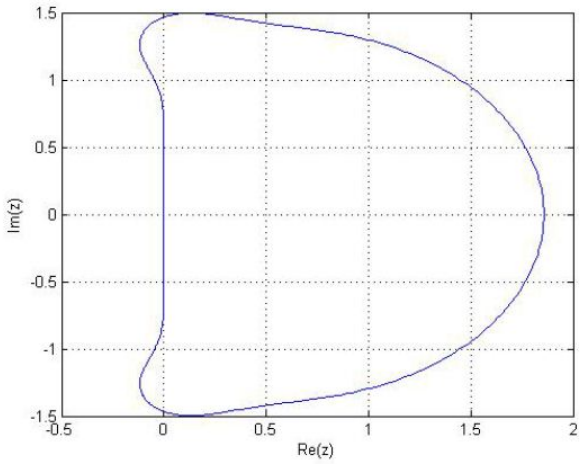

Fig. 3. The region of absolute stability for Scheme (2.19)

4 Implementation and Conclusion

4.1 Implementation

We test the strength of these block methods by solving some non-linear systems of first Order ODEs.

Problem 1

$$y' = \begin{bmatrix} -10004y_1 & 10000y_2^4 \\ y_1 & -y_2(1+y_2^3) \end{bmatrix}$$

$$y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} h = 0.1, x \in [0, 20]$$

Exact solution

$$y(x) = \begin{bmatrix} e^{-4x} \\ e^{-x} \end{bmatrix}$$

Problem 2: Lotka Volterra equation

$$y' = \begin{bmatrix} 1.2y_1 & -0.6y_1y_2 \\ -0.8y_2 & 0.3y_1y_2 \end{bmatrix}$$

$$y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, h = 0.001, x \in [0, 20]$$

Problem 3: Robertson chemical equation

$$y_1' = -0.04y_1 + 10000y_2y_3$$

$$y_2' = 0.04y_1 - 10000y_2y_3 - 3.0 \times 10^7 y_2^2$$

$$y_3' = 3.0 \times 10^7 y_2^2$$

$$y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, 0 \leq x \leq 400, h = 0.0001$$

Solving Problem 1, 2 and 3, we used MATLAB software to solve the non-linear systems. The solution curve using (2.11), (2.15) and (2.19) are given in Fig 4, Fig 5 and Fig 6. respectively.

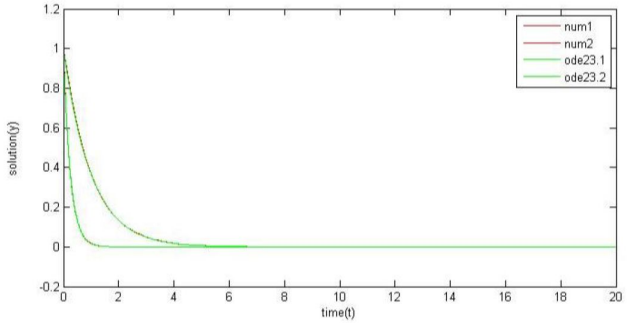

Fig. 4. The solution curve of problem 1 with Scheme (2.11)

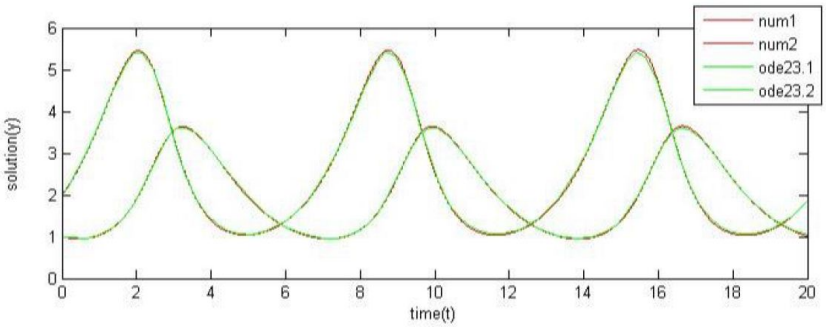

Fig. 5. The solution curve of problem 2 with Scheme (2.15)

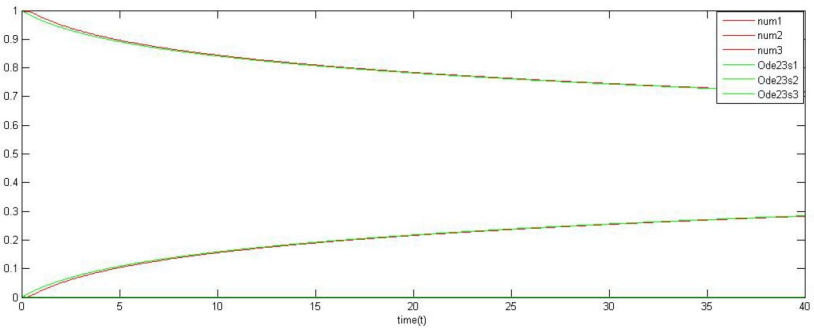

Fig. 6. The solution curve of problem 3 with Scheme (2.19)

4.2 Solutions of Scheme (2.11) and Scheme (2.15) for Problem 1

Table 2. Numerical and Exact solutions of problem 1 using Scheme (2.11) and (2.15)

x	Exact1	Exact2	Numerical1		Numerical2	
			Scheme(2.11)	Scheme(2.15)	Scheme(2.11)	Scheme(2.15)
0.1	6.703E-01	9.048E-01	6.196E-01	6.195E-01	9.049E-01	9.048E-01
0.2	4.493E-01	8.187E-01	4.153E-01	4.152E-01	8.188E-01	8.187E-01
0.3	3.012E-01	7.408E-01	2.784E-01	2.783E-01	7.409E-01	7.408E-01
0.4	2.019E-01	6.703E-01	1.866E-01	1.866E-01	6.704E-01	6.703E-01
0.5	1.353E-01	6.065E-01	1.251E-01	1.251E-01	6.066E-01	6.065E-01
0.6	9.070E-02	5.488E-01	8.390E-02	8.380E-02	5.489E-01	5.488E-01
0.7	6.080E-02	4.966E-01	5.620E-02	5.620E-02	4.966E-01	4.966E-01
0.8	4.080E-02	4.493E-01	3.770E-02	3.770E-02	4.494E-01	4.493E-01
0.9	2.730E-02	4.066E-01	2.530E-02	2.530E-02	4.066E-01	4.066E-01
1.0	1.830E-02	3.679E-01	1.690E-02	1.690E-02	3.679E-01	3.679E-01
1.1	1.230E-02	3.329E-01	1.140E-02	1.130E-02	3.329E-01	3.329E-01
1.2	8.200E-03	3.012E-01	7.600E-03	7.600E-03	3.013E-01	3.012E-01
1.3	5.500E-03	2.725E-01	5.100E-03	5.100E-03	2.726E-01	2.725E-01
1.4	3.700E-03	2.466E-01	3.400E-03	3.400E-03	2.467E-01	2.466E-01
1.5	2.500E-03	2.231E-01	2.300E-03	2.300E-03	2.232E-01	2.231E-01
1.6	1.700E-03	2.019E-01	1.500E-03	1.500E-03	2.020E-01	2.019E-01
1.7	1.100E-03	1.827E-01	1.000E-03	1.000E-03	1.827E-01	1.827E-01
1.8	7.000E-04	1.653E-01	7.000E-04	7.000E-04	1.654E-01	1.653E-01
1.9	5.000E-04	1.496E-01	5.000E-04	5.000E-04	1.496E-01	1.496E-01
2.0	3.000E-04	1.353E-01	3.000E-04	3.000E-04	1.354E-01	1.353E-01

Comparing solutions of Scheme (2.11) and Scheme (2.15) with the analytical solutions, we deduce the maximum error.

Maximum error(Scheme (2.11))= $y_1 = 1.000E-15, y_2 = 4.354E-07$

Maximum error(Scheme (2,15))= $y_1 = 1.000E-15, y_2 = 4.768E-08$

4.3 Comparing solutions of Scheme (2.19) with those of Butcher and Hojjati [5] for problem 3 (Robertson Chemical Equation)

Table 3. Comparison of Scheme (2.19) with Butcher and Hojjati [5]

x	Scheme (2.19)	Butcher and Hojjati [5]
0.4	9.851721137972580E-01	9.851721138620630E-01
	3.386395378900000E-05	3.386395379595400E-05
	1.479402218444700E-02	1.479402213590220E-02
4	9.055186779860960E-01	9.055186784344190E-01
	2.240475687100000E-05	2.240475693804370E-05
	9.445891660335700E-02	9.445891599170860E-02
40	7.158270638024940E-01	7.158270698910200E-01
	9.185534750000000E-06	9.185534641631410E-06
	2.841637441465430E-01	2.841637507954150E-01
400	4.505186352019710E-01	4.505186908340870E-01
	3.222901389000000E-06	3.222901061260970E-06
	5.494780766336790E-01	5.494782035239040E-01

There is a fair agreement between the results of Scheme(2.19) and that of Butcher and Hojjati [5]. However the method derived by Butcher and Hojjati [5] is second derivative method which should have an advantage over Scheme (2.19).

5 Conclusions

The methods we derived competes quite well with the in-built ode23 solvers in Matlab. The solution curves show the performance of the Block Hybrid Explicit Integrators for step numbers $k = 2, 3, 4$. These methods are shown to be $A(\alpha)$-stable and convergent. Therefore suitable for solution of non-linear system of first order ODEs.

Competing Interests

All authors have declared that there's no competing interest.

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