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Solving Systems of Fractional Differential Equations Using Sumudu Transform Method

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Authors' contributions

This work was carried out in collaboration between all authors. Authors YAA and AMSM designed the study, performed the numerical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors AMSM and ESMY managed the analyses of the study. Author AMSM managed the literature searches. All authors read and approved the final manuscript.

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Review Article

Abstract

In this paper we are interested in showing the approximate analytical solutions for systems of fractional differential equations and nonlinear biochemical reaction model by using Sumudu transform method. The fractional derivatives are described in the Caputo sense. The applications related to Sumudu transform method have been developed for differential equations to the ex- tent of access to approximate analytical solutions of systems of fractional differential equations. The solutions of our model equations are calculated in the form of convergent series with easily computable components. Some examples are solved as illustrations, using symbolic computation. The numerical results show that the approach is easy to implement and accurate when applied to systems of fractional differential equations. The method introduce a promising tool for solving many linear and nonlinear fractional differential equations.

Keywords: Caputo derivative; Sumudu transform; Predator-prey model; biochemical model.

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1 Introduction

In this study, we consider the system of fractional differential equations:

$$\begin{aligned}
D_*^{\alpha_1} x_1(t) &= f_1(t,, x_1, x_2, \ldots\ldots, x_n) \\
D_*^{\alpha_2} x_1(t) &= f_2(t,, x_1, x_2, \ldots\ldots, x_n) \\
&\vdots \\
D_*^{\alpha_n} x_1(t) &= f_2(t,, x_1, x_2, \ldots\ldots, x_n)
\end{aligned} \tag{1}$$

Where $D_*^{\alpha_i}$ is the derivative of $x_i$of order $\alpha_i$in the sense of Caputo and $0<\alpha_i\leq 1$, subject to the initial conditions

$$x_1(0) = c_1, \quad x_2(0) = c_2, \ldots, x_n(0) = c_n \tag{2}$$

In the 19th century, in 1913, Michael and Manten gave a simple idea to describe enzyme processes, and the basic enzymatic model was given by the planner [1,2].

$$M + N \rightleftharpoons \mathrm{L} \rightarrow \mathrm{M} + \mathrm{R} \tag{3}$$

where M is the enzyme, N the substrate, L the enzyme-substrate intermediate complex and R the product, from the law of mass action, which states that reaction rates are proportional to the concentrations of the reactants, the time evolution of the scheme Eq. (3) can be determined from the solution of the system of coupled nonlinear ordinary differential equations [2,3].

$$\frac{dN}{dt} = -k_1 MN + k_{-1} L, \tag{4}$$

$$\frac{dM}{dt} = -k_1 MN + (k_{-1} + k_2) L, \tag{5}$$

$$\frac{dL}{dt} = k_1 MN - (k_{-1} + k_2) L, \tag{6}$$

$$\frac{dR}{dt} = k_2 L \tag{7}$$

With initial conditions

$$\mathrm{N}(0) = N_0, \quad \mathrm{M}(0) = M_0, \quad \mathrm{L}(0) = 0, \quad \mathrm{R}(0) = 0 \tag{8}$$

Where the parameters k_1, k_{-1} and k_2 are positive rate constants for each reaction.Systems Eqs from (4) to (7) can beshortened to only two equations for N and L and in dimensionless form of concentrations of substrate,u, and intermediate complex between enzyme and substrate, v, are given by [2,3].

$$\frac{du}{dt} = -u + (\beta - \alpha) v + uv, \tag{9}$$

$$\frac{dv}{dt} = \frac{1}{\gamma}(u - \beta v - uv) \tag{10}$$

subject to the initial conditions

$$\mathrm{u}(0) = 1, \quad \mathrm{v}(0) = 0 \tag{11}$$

Where α,β, and γ are dimensionless parameters.

Several studies are of interested in differential equations of fractional order due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. Recently, a large amount of literatures developed concerning the application of fractional differential equations in nonlinear dynamics [4-15]. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, they are used extensively. Recently, the Adomian decomposition method and variational iteration method have been used for solving a wide range of problems [9,10,16-24]. The two methods were used in a direct way without using linearization, perturbation or restrictive assumptions.

There are numerous integral transforms such as the Laplace, Sumudu, Fourier, Mellin, and Hankel to solve PDEs. Of these, the Laplace transformation and Sumudu transformation are the most widely used. The Sumudu transformation method is one of the most important transform methods introduced in the early 1990s by Gamage K. Watugala. It is a powerful tool for solving many kinds of PDEs in various fields of science and engineering. And also various methods are combined with the Sumudu transformation method such as the homotopy Analysis Sumudu Transform Method (HASTM) [11-15,25-30] which is a combination of the homotopy analysis method and the Sumudu transformation method. The objective of the present paper is to extend the application of the homotopy perturbation sumudu transform method to provides an approximate solutions for initial value problems of linear and nonlinear fractional differential equations and to make comarison with that obtained by Adomian decomposition method [4,31-34], where the fractional derivative is considered in Caputo sense.

The paper is structured in six sections. In section 2, we begin with an introduction to some necessary definitions of fractional calculus theory. In section 3 we describe the homotopy perturbation sumudu transform method. In section 4, we present three examples to show the efficiency of using HPSTM to solve FDEs and also to compare our result with those obtained by other existing methods. Finally, relevant conclusions are drawn in section 5.

2 Basic Definitions of Fractional Calculus

In this section, we present the basic definitions and properties of the fractional calculus theory, which are used further in this paper.

Definition 1 A real function f(t), t>0, is said to be in the space C_α , $\alpha \in R$, if there exists a real number p>α such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0,\infty)$, and it is said to be in the space C_α^m if $f^m \in C_\alpha$, m∈N.

Definition 2 The Riemann-Liouville fractional integral operator of order α>0,for t>0 is defined as [8,35]:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\,\xi \tag{12}$$

$$J^0 f(t) = f(t) \tag{13}$$

The Riemann-liouville derivative has certain disadvantage when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by M. Caputo in his work on the theory of viscoelasticity [35].

Properties of the operator J^αcan be found in where α>0, t>0 and Γ(α) is the gamma function.

Definition 3 The Caputo fractional derivative of f(t) of order α>0 with t>0 is defined as [8,36,37].

$$D_*^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} f^{(m)}(\xi) d\,\xi \tag{14}$$

For m-1<α≤m, m∈N, f(t)∈C_α^m.

Definition 4 The Sumudu transform is defined over the set of functions [4,11-15,31-34].

$$A = \left\{ f(t) \,\middle|\, \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}} if\ t\epsilon(-1)^j \times [0,\infty) \right\} \tag{15}$$

by the following formula:

$$\overline{f}(u) = S[f(t)] = \int_0^\infty f(ut) e^{-t} dt \tag{16}$$

where u∈ (τ_1, τ_2)

Some special properties of the Sumudu transform are as follows [38]

1. S[1]=1;
2. $S\left[\frac{t^m}{\Gamma(m+1)}\right] = u^m$; $m > 0$

Definition 5 The Sumudu transform of Caputo fractional derivative is defined as follows [38]:

$$S[D_t^\alpha f(t)] = u^\alpha S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0), \;\; m-1 < \alpha \le m. \tag{17}$$

Theorem ([11],[32])

$$\boldsymbol{S}\left[f^{(n)}(t)\right] = u^{-n}\left[f(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right] \qquad for\;\; n \ge 1$$

At very special case for n=1

$$\boldsymbol{S}\left[f^{(1)}(t)\right] = \frac{\boldsymbol{1}}{\boldsymbol{u}}[\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{0})].$$

This theorem is very important to calculate approximate solution of the problems.

3 The Homotopy Perturbation Sumudu Transform Method

In order to elucidate the solution procedure of this method, we consider a general fractional nonlinear differential equation of the form [11-15, 25-30]:

$$D_*^\alpha x(t) + Lx(t) + Nx(t) = q(t) \tag{18}$$

with m-1<α≤m, and subject to the initial condition

$$x^j(0) = c_j, \quad\;\; j = 0,1, \dots \dots, m-1, \tag{19}$$

Where $D_*^\alpha x(t)$is the Caputo fractional derivative, q(t) is the source term, L is the linear operator and N is the general nonlinear operator.

Applying the Sumudu transform (denoted throughout this paper by S) on both sides of Eq.(18), we have

$$S[D_*^\alpha x(t)] + S[Lx(t)] + S[Nx(t)] = S[q(t)] \tag{20}$$

Using the property of the Sumudu transform and the initial conditions in Eq.(19), we have

$$S[x(t)] = \sum_{k=0}^{m-1} u^{-\alpha+k} x^{(k)}(0) + u^\alpha S[q(t)] - u^\alpha S[Lx(t) + Nx(t)], \tag{21}$$

Operating with the Sumudu inverse on both sides of Eq.(21) we get

$$x(t) = G(t) - S^{-1}\big[u^\alpha S[Lx(t) + Nx(t)]\big] \tag{22}$$

Where G(t) represents the term arising from the source term and the prescribed initial conditions. Now, applying the classical perturbation technique. And assuming that the solution of Eq.(22) is in the form

$$x(t) = \sum_{m=0}^{\infty} p^m x_m(t) \tag{23}$$

where p∈[0,1] is the homotopy parameter. The nonlinear term of Eq.(22) can be decomposed as

$$Nx(t) = \sum_{m=0}^{\infty} p^m A_m(t). \tag{24}$$

forsome Adomian's polynomials A_m, which can be calculated with the formula [39]

$$A_m = \frac{1}{m!}\frac{d^m}{dp^m}\left[N\left(\sum_{i=0}^{\infty} p^i x_i(t)\right)\right]_{p=0}, \quad m = 0,1,2,\dots\dots \tag{25}$$

Substituting Eq.(23) and (25) in Eq.(22), we get

$$\sum_{m=0}^{\infty} p^m x_m(t) = G(t) - S^{-1}\left[u^\alpha S\left[L\left(\sum_{m=0}^{\infty} p^m x_m(t)\right) + \sum_{m=0}^{\infty} p^m A_m\right]\right]. \tag{26}$$

Equating the terms with identical powers of p, we can obtain a series of equations as the follows:

$$\begin{aligned} p^0 &: x_0(t) = G(t) \\ p^1 &: x_1(t) = -S^{-1}\big[u^\alpha S[Lx_0(t) + A_0]\big] \\ p^2 &: x_2(t) = -S^{-1}\big[u^\alpha S[Lx_1(t) + A_1]\big] \\ p^3 &: x_3(t) = -S^{-1}\big[u^\alpha S[Lx_2(t) + A_2]\big] \\ &\vdots \end{aligned} \tag{27}$$

Finally, we approximate the analytical solution x(t) by truncated series as

$$x(t) = \lim_{M\to\infty} \sum_{m=0}^{M} p^m x_m(t). \tag{28}$$

4 Applications

To demonstrate the effectiveness of the proposed algorithm, two special cases of the system of fractional differential equations [1,2] and one case of Nonlinear biochemical reaction model equations [9-11] will be studied. All the results are calculated by using the symbolic calculus software Mathematica.

Example 1 Let us consider the following system of two linear fractional differential equations [6]

$$\begin{cases} D_*^{\alpha} x(t) = x(t) + y(t) \\ D_*^{\beta} y(t) = -x(t) + y(t) \end{cases} \tag{29}$$

where t>0, 0<α, β≤1, subject to the initial conditions

$$\mathrm{x}(0) = 0, \quad \mathrm{y}(0) = 1 \tag{30}$$

Taking the Sumudu transform on both sides of Eq.(29), thus we get

$$\begin{cases} S[D_*^{\alpha} x(t)] = S[x(t) + y(t)]\,, \\ S\left[D_*^{\beta} y(t)\right] = S[-x(t) + y(t)] \end{cases} \tag{31}$$

Using the property of the Sumudu transform and the initial condition in Eq.(30), we have

$$\begin{cases} u^{-\alpha} S[x(t)] = u^{-\alpha} x(0) + S[x(t) + y(t)] \\ u^{-\beta} S[y(t)] = u^{-\beta} y(0) + S[-x(t) + y(t)] \end{cases} \tag{32}$$

Operating with the Sumudu inverse on both sides of Eq.(32) we get

$$\begin{cases} x(t) = S^{-1}\left[u^{\alpha} S[x(t) + y(t)]\right], \\ y(t) = 1 + S^{-1}\left[u^{\beta} S[-x(t) + y(t)]\right] \end{cases} \tag{33}$$

By applying the homotopy perturbation method, and substituting Eq.(23) in Eq.(33) we have

$$\sum_{m=0}^{\infty} p^m x_m(t) = S^{-1}\left[u^{\alpha} S\left[\sum_{m=0}^{\infty} p^m x_m(t) + \sum_{m=0}^{\infty} p^m y_m(t)\right]\right] \tag{34}$$

and

$$\sum_{m=0}^{\infty} p^m y_m(t) = 1 + S^{-1}\left[u^{\beta} S\left[-\sum_{m=0}^{\infty} p^m x_m(t) + \sum_{m=0}^{\infty} p^m y_m(t)\right]\right] \tag{35}$$

Equating the terms with identical powers of p, we get

$$p^0: \begin{cases} x_0(t) = 0 \\ y_0(t) = 1 \end{cases}$$

$$p^1: \begin{cases} x_1(t) = \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \\ y_1(t) = \dfrac{t^{\beta}}{\Gamma(\beta+1)} \end{cases}$$

$$p^2: \begin{cases} x_2(t) = \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ y_2(t) = \dfrac{t^{2\beta}}{\Gamma(2\beta+1)} - \dfrac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \end{cases}$$

$$p^3:\begin{cases} x_3(t) = \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dfrac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} \\ y_3(t) = \dfrac{t^{3\beta}}{\Gamma(3\beta+1)} - \dfrac{2t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} - \dfrac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \end{cases}$$

$$p^4:\begin{cases} x_4(t) = \dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \dfrac{t^{2\alpha+2\beta}}{\Gamma(2\alpha+2\beta+1)} + \dfrac{t^{\alpha+3\beta}}{\Gamma(\alpha+3\beta+1)} - \dfrac{t^{3\alpha+\beta}}{\Gamma(3\alpha+\beta+1)} \\ y_4(t) = \dfrac{t^{4\beta}}{\Gamma(4\beta+1)} - \dfrac{3t^{\alpha+3\beta}}{\Gamma(\alpha+3\beta+1)} - \dfrac{t^{2\alpha+2\beta}}{\Gamma(2\alpha+2\beta+1)} - \dfrac{t^{3\alpha+\beta}}{\Gamma(3\alpha+\beta+1)} \end{cases}$$

$$p^5:\begin{cases} x_5(t) = \dfrac{t^{5\alpha}}{\Gamma(5\alpha+1)} - \dfrac{2t^{3\alpha+2\beta}}{\Gamma(3\alpha+2\beta+1)} - \dfrac{2t^{2\alpha+3\beta}}{\Gamma(2\alpha+3\beta+1)} - \dfrac{2t^{4\alpha+\beta}}{\Gamma(4\alpha+\beta+1)} + \dfrac{t^{\alpha+4\beta}}{\Gamma(\alpha+4\beta+1)} \\ y_5(t) = \dfrac{t^{5\beta}}{\Gamma(5\beta+1)} - \dfrac{4t^{\alpha+4\beta}}{\Gamma(\alpha+4\beta+1)} - \dfrac{t^{4\alpha+\beta}}{\Gamma(4\alpha+\beta+1)} \end{cases}$$

$$\vdots \qquad (36)$$

If we α→1 in Eq.(36) or solve Eq.(29) and (30) with α=1,β=1, we obtain

$$\begin{cases} x(t) = t + t^2 + \dfrac{t^3}{3} - \dfrac{t^5}{30} + \cdots \\ y(t) = 1 + t - \dfrac{t^3}{3} - \dfrac{t^4}{6} - \dfrac{t^5}{30} + \cdots \end{cases}$$

Solve Eq.(29) and (30) with α=0.7,β=0.9, we obtain

$$\begin{cases} x(t) = \dfrac{t^{\frac{7}{10}}}{\Gamma\left(\frac{17}{10}\right)} + \dfrac{t^{\frac{7}{5}}}{\Gamma\left(\frac{12}{5}\right)} + \dfrac{t^{\frac{8}{5}}}{\Gamma\left(\frac{13}{5}\right)} + \dfrac{t^{\frac{21}{10}}}{\Gamma\left(\frac{31}{10}\right)} + \dfrac{8t^{\frac{5}{2}}}{15\Gamma\left(\frac{1}{2}\right)} + \dfrac{t^{\frac{14}{5}}}{\Gamma\left(\frac{19}{5}\right)} - \dfrac{t^{\frac{16}{5}}}{\Gamma\left(\frac{21}{5}\right)} + \dfrac{t^{\frac{17}{5}}}{\Gamma\left(\frac{22}{5}\right)} - \dfrac{t^3}{6} + \dfrac{16t^{\frac{7}{2}}}{\Gamma\left(\frac{1}{2}\right)} - \dfrac{2t^{\frac{39}{10}}}{\Gamma\left(\frac{49}{10}\right)} - \dfrac{2t^{\frac{41}{10}}}{\Gamma\left(\frac{47}{10}\right)} + \dfrac{t^{\frac{43}{10}}}{\Gamma\left(\frac{53}{10}\right)} + \cdots \\ y(t) = 1 + \dfrac{t^{\frac{9}{10}}}{\Gamma\left(\frac{19}{10}\right)} + \dfrac{t^{\frac{9}{5}}}{\Gamma\left(\frac{14}{5}\right)} - \dfrac{t^{\frac{8}{5}}}{\Gamma\left(\frac{13}{5}\right)} + \dfrac{t^{\frac{27}{10}}}{\Gamma\left(\frac{37}{10}\right)} - \dfrac{16t^{\frac{5}{2}}}{15\Gamma\left(\frac{1}{2}\right)} - \dfrac{t^{\frac{23}{10}}}{\Gamma\left(\frac{33}{10}\right)} + \dfrac{t^{\frac{18}{5}}}{\Gamma\left(\frac{23}{5}\right)} - \dfrac{3t^{\frac{17}{5}}}{\Gamma\left(\frac{22}{5}\right)} - \dfrac{t^{\frac{16}{5}}}{\Gamma\left(\frac{21}{5}\right)} - \dfrac{t^3}{6} + \cdots \end{cases}$$

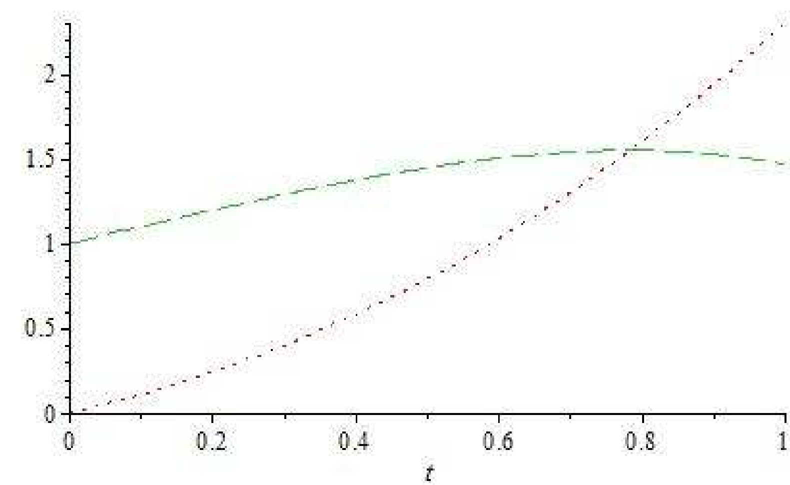

Fig. 1. The behavior of x(t) and y(t) at $\alpha = 1,\ \beta = 1$.

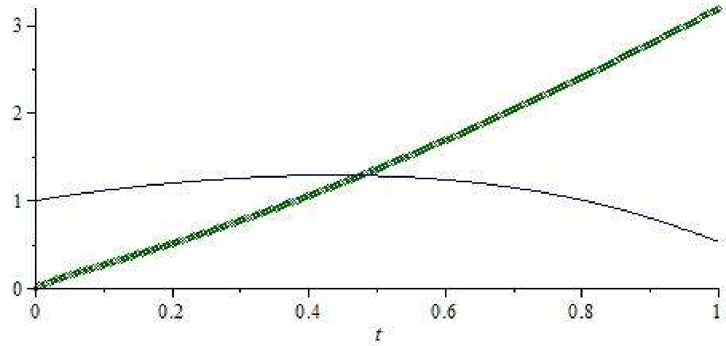

Fig. 2. The behavior of x(t) and y(t) at $\alpha = 0.7, \beta = 0.9$.

It is evident that the efficiency of this approach can dramatically enhanced by computing further terms of x(t), y(t) when the Homotopy Perturbation Sumudu Transform Method is used. The results in Fig. 1 and Fig. 2 are in full agreement with the results obtained in [6] using the Adomain decomposition method.

Example 2 Lastly we consider the following system of three nonlinear fractional differential equations:[6]

$$\begin{cases} D_*^{\alpha} x(t) = 2y^2(t) \\ D_*^{\beta} y(t) = t x(t) \\ D_*^{\gamma} z(t) = y(t) z(t) \end{cases} \tag{37}$$

where t>0, 0<α, β, γ≤1, subject to the initial conditions

$$x(0) = 0, \quad y(0) = 1, \quad z(0) = 1 \tag{38}$$

Taking the Sumudu transform on both sides of Eq.(37), thus we get

$$\begin{cases} S[D_*^{\alpha} x(t)] = S[2y^2(t)] \\ S\left[D_*^{\beta} y(t)\right] = S[t x(t)] \\ S\left[D_*^{\gamma} z(t)\right] = S[y(t). z(t)] \end{cases} \tag{39}$$

Using the property of the Sumudu transform and the initial condition in Eq.(38), we have

$$\begin{cases} u^{-\alpha} S[x(t)] = u^{-\alpha} x(0) + S[2y^2(t)] \\ u^{-\beta} S[y(t)] = u^{-\beta} y(0) + S[t x(t)] \\ u^{-\gamma} S[z(t)] = u^{-\gamma} z(0) + S[y(t). z(t)] \end{cases} \tag{40}$$

Operating with the Sumudu inverse on both sides of Eq.(40) we get

$$\begin{cases} x(t) = S^{-1}\left[u^{\alpha} S[2y^2(t)]\right] \\ y(t) = 1 + S^{-1}\left[u^{\beta} S[t x(t)]\right] \\ z(t) = 1 + S^{-1}\left[u^{\gamma} S[y(t). z(t)]\right] \end{cases} \tag{41}$$

By applying the homotopy perturbation method, and substituting Eq.(23) in Eq.(41) we have

$$\begin{cases} \sum_{m=0}^{\infty} p^m x_m(t) = S^{-1}\left[u^{\alpha} S\left[2\left(\sum_{m=0}^{\infty} p^m y_m(t)\right)^2\right]\right] \\ \sum_{m=0}^{\infty} p^m y_m(t) = 1 + S^{-1}\left[u^{\beta} S\left[t \sum_{m=0}^{\infty} p^m x_m(t)\right]\right] \\ \sum_{m=0}^{\infty} p^m z_m(t) = 1 + S^{-1}\left[u^{\gamma} S\left[\left(\sum_{m=0}^{\infty} p^m y_m(t)\right).\left(\sum_{m=0}^{\infty} p^m z_m(t)\right)\right]\right] \end{cases} \tag{42}$$

Equating the terms with identical powers of p, we get

$$p^0: \begin{cases} x_0(t) = 0 \\ y_0(t) = 1 \\ z_0(t) = 1 \end{cases}$$

$$p^1: \begin{cases} x_1(t) = \dfrac{2t^{\alpha}}{\Gamma(\alpha+1)} \\ y_1(t) = 0 \\ z_1(t) = \dfrac{t^{\gamma}}{\Gamma(\gamma+1)} \end{cases}$$

$$p^2: \begin{cases} x_2(t) = 0 \\ y_2(t) = \dfrac{2(\alpha+1)t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} \\ z_2(t) = \dfrac{t^{2\gamma}}{\Gamma(2\gamma+1)} \end{cases}$$

$$p^3: \begin{cases} x_3(t) = \dfrac{8(\alpha+1)t^{2\alpha+\beta+1}}{\Gamma(2\alpha+\beta+2)} + \dfrac{8(\alpha+1)^2\Gamma(2\alpha+2\beta+3)t^{3\alpha+2\beta+2}}{\Gamma(2\alpha+\beta+2)\Gamma(3\alpha+2\beta+3)} \\ y_3(t) = 0 \\ z_3(t) = \dfrac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \dfrac{2(\alpha+1)t^{\alpha+\beta+\gamma+1}}{\Gamma(\alpha+\beta+\gamma+2)} + \dfrac{2(\alpha+1)\Gamma(\alpha+\beta+\gamma+2)t^{\alpha+\beta+2\gamma+1}}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+2)\Gamma(\alpha+\beta+2\gamma+2)} + \\ \dfrac{2(\alpha+1)\Gamma(\alpha+\beta+2\gamma+2)t^{\alpha+\beta+3\gamma+1}}{\Gamma(2\gamma+1)\Gamma(\alpha+\beta+2)\Gamma(\alpha+\beta+3\gamma+2)} \end{cases}$$

$$p^4: \begin{cases} x_4(t) = 0 \\ y_4(t) = \dfrac{8(\alpha+1)\Gamma(2\alpha+\beta+3)t^{2\alpha+2\beta+2}}{\Gamma(2\alpha+\beta+2)\Gamma(2\alpha+2\beta+3)} + \dfrac{8(\alpha+1)^2\Gamma(2\alpha+2\beta+3)\Gamma(3\alpha+2\beta+4)t^{3\alpha+3\beta+3}}{\Gamma(2\alpha+\beta+2)\Gamma(3\alpha+2\beta+3)\Gamma(3\alpha+3\beta+4)} \\ z_4(t) = \dfrac{t^{4\gamma}}{\Gamma(4\gamma+1)} + \dfrac{2(\alpha+1)t^{\alpha+\beta+2\gamma+1}}{\Gamma(2\alpha+2\beta+2)} + \dfrac{2(\alpha+1)\Gamma(\alpha+\beta+\gamma+2)t^{\alpha+\beta+3\gamma+1}}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+2)\Gamma(\alpha+\beta+3\gamma+2)} + \\ \dfrac{2(\alpha+1)\left[\left((\Gamma(3\gamma+1)\Gamma(\alpha+\beta+2\gamma+2)+(\Gamma(2\gamma+1)\Gamma(\alpha+\beta+3\gamma+2)\right)\right]t^{\alpha+\beta+4\gamma+1}}{\Gamma(2\gamma+1)\Gamma(3\gamma+1)\Gamma(\alpha+\beta+2)\Gamma(\alpha+\beta+4\gamma+2)} + \dfrac{4(\alpha+1)^2\Gamma(\alpha+\beta+\gamma+2)t^{2\alpha+2\beta+2\gamma+2}}{\Gamma(\alpha+\beta+2)\Gamma(\alpha+\beta+\gamma+2)\Gamma(2\alpha+2\beta+2\gamma+3)} + \\ \dfrac{4(\alpha+1)^2\Gamma(\alpha+\beta+\gamma+2)\Gamma(2\alpha+2\beta+2\gamma+3)\,t^{2\alpha+2\beta+3\gamma+2}}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+2)^2\Gamma(\alpha+\beta+2\gamma+2)\Gamma(2\alpha+2\beta+3\gamma+3)} + \dfrac{4(\alpha+1)^2\Gamma(\alpha+\beta+2\gamma+2)\Gamma(2\alpha+2\beta+3\gamma+3)\,t^{2\alpha+2\beta+4\gamma+2}}{\Gamma(2\gamma+1)\Gamma(\alpha+\beta+2)^2\Gamma(\alpha+\beta+3\gamma+2)\Gamma(2\alpha+2\beta+4\gamma+3)} \end{cases}$$

$$\vdots$$

(43)

If we α→1 in Eq.(43) or solve Eq.(37) and (38) with α=β=γ=1, we obtain

$$\begin{cases} x(t) = 2t + \frac{2t^4}{3} + \frac{4t^7}{21} + \cdots \\ y(t) = 1 + \frac{2t^3}{3} + \frac{t^6}{9} + \frac{4t^9}{189} + \cdots \\ z(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{5t^4}{24} + \frac{t^5}{6} + \frac{7t^6}{90} + \cdots \end{cases}$$

Solve Eq.(37) and (38) with α=0.5, β=0.4, γ=0.3 we obtain

$$\begin{cases} x(t) = \frac{4\sqrt{t}}{\sqrt{\pi}} + \frac{12t^{12/5}}{\Gamma\left(\frac{17}{5}\right)} + \cdots \\ y(t) = 1 + \frac{3t^{19/10}}{\Gamma\left(\frac{29}{10}\right)} + \cdots \\ z(t) = 1 + \frac{t^{3/10}}{\Gamma\left(\frac{13}{10}\right)} + \frac{t^{3/5}}{\Gamma\left(\frac{8}{5}\right)} + \frac{t^{9/5}}{\Gamma\left(\frac{19}{10}\right)} + \frac{3t^{11/5}}{\Gamma\left(\frac{16}{5}\right)} + \frac{15t^{14/5}}{15\Gamma\left(\frac{19}{5}\right)} + \frac{t^{6/5}}{\Gamma\left(\frac{11}{5}\right)} + \cdots \end{cases}$$

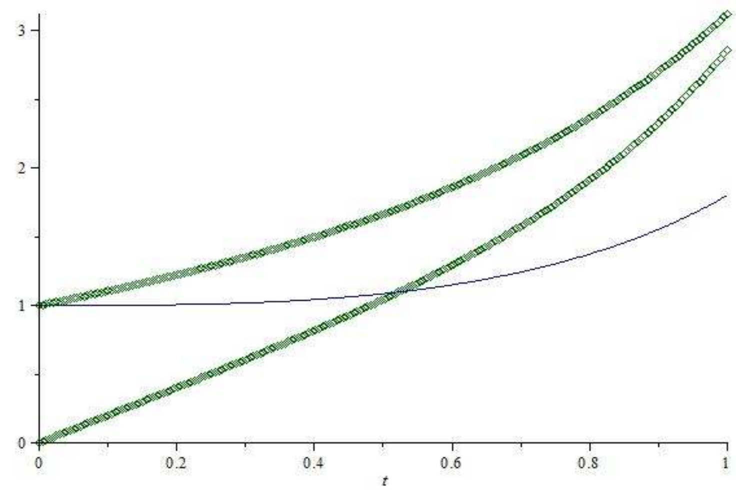

Fig. 3. The behavior of x(t), y(t) and z(t) at $\alpha = 0.5$, $\beta = 0.4$, $\gamma = 0.3$.

It is evident that the efficiency of this approach can dramatically enhanced by computing further terms of x(t), y(t) and z(t)when the Homotopy Perturbation Sumudu Transform Method is used the results in Fig.3 is in full agreement with the results obtained in [7] using the Adomain decomposition method.

Example 3 Let us consider the following system of the nonlinear biochemical reaction model as [29,30].

$$\frac{du}{dt} = -u + (\beta - \alpha)v + uv, \tag{44}$$

$$\frac{dv}{dt} = \frac{1}{\gamma}(u - \beta v - uv) \tag{45}$$

subject to the initial conditions

$$\mathrm{u}(0)=1, \quad \mathrm{v}(0)=0 \tag{46}$$

First by taken the sumudu transform to Eqs (44) and Eqs (45) as:

$$\begin{cases} S[\dot{u}] = \frac{S[u(t)]-u(0)}{u} = S[-u+(\beta-\alpha)v+uv] \\ S[\dot{v}] = \frac{S[v(t)]-v(0)}{u} = S\left[\frac{1}{\gamma}(u-\beta v-uv)\right] \end{cases} \tag{47}$$

Second by taken the inverse of sumudu transform to the Eq. (47) with the initial condition we have:

$$\begin{cases} u(t) = 1 + s^{-1}\big[uS[-u+(\beta-\alpha)v+uv]\big] \\ v(t) = s^{-1}\left[uS\left[\frac{1}{\gamma}(u-\beta v-uv)\right]\right] \end{cases} \tag{48}$$

Third by assuming that the solution as infinite series of unknown functions:

$$\begin{cases} \sum_{n=0}^{\infty} u_n(t) = 1 + s^{-1}\left[uS\left[-\sum_{n=0}^{\infty} u_n(t) + (\beta-\alpha)\sum_{n=0}^{\infty} v_n(t) + A_n\right]\right] \\ \sum_{n=0}^{\infty} v_n(t) = s^{-1}\left[uS\left[\frac{1}{\gamma}\left(\sum_{n=0}^{\infty} u_n(t) - \beta\sum_{n=0}^{\infty} v_n(t) - A_n\right)\right]\right] \end{cases} \tag{49}$$

Where $A_n$are Adomian polynomials that refers to the nonlinear term and the first three components of the Adomian polynomials are given as follows:

$A_0 = u_0 v_0, \qquad A_1 = u_0 v_1 + u_1 v_0, \qquad A_2 = u_0 v_2 + u_1 v_1 + u_2 v_0$

then we have

$$u_0 = 1$$

$$u_1 = s^{-1}\big[uS[-u_0+(\beta-\alpha)v_0+A_0]\big]$$

$$u_{k+1} = s^{-1}\big[uS[-u_k+(\beta-\alpha)v_k+A_k]\big]$$

and

$$v_0 = 0$$
$$v_1 = s^{-1}\left[uS\left[\frac{1}{\gamma}(u_0-\beta v_0-A_0)\right]\right]$$
$$v_{k+1} = s^{-1}\left[uS\left[\frac{1}{\gamma}(u_k-\beta v_k-A_k)\right]\right]$$

by using that α=0.375, β=1, γ=0.1 we have

$$A_0 = 0$$

$$u_1 = s^{-1}\big[uS[-1+(\beta-\alpha)0+0]\big] = -t$$

$$v_1 = s^{-1}\left[uS\left[\frac{1}{0.1}(1-\beta*0-0)\right]\right] = 10t$$
$$A_1 = 10t$$

$$u_2 = 2.375t^2$$
$$v_2 = -10t^2$$

$$A_2 = -11t^2$$

$$u_3 = -17.25t^3$$

$$v_3 = 741.25t^3$$

$$\vdots$$

By continue we get the solution as series:

$$u(t) = 1 - t + 2.375t^2 - 17.25t^3 + \cdots \quad (50)$$

$$v(t) = 10t - 105t^2 + 741.25t^3 + \cdots \quad (51)$$

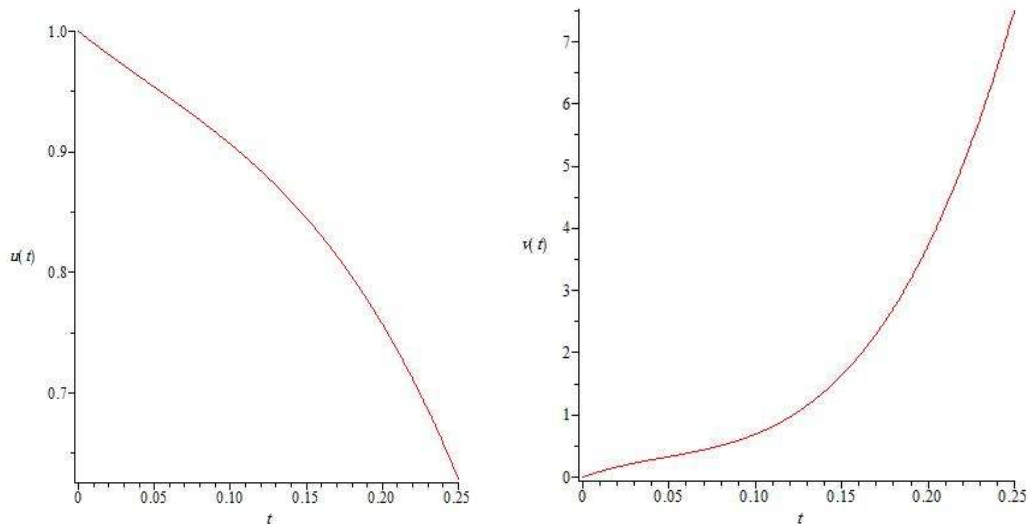

Fig. 4. The solutions of u(t) and v(t) at $\alpha = 0.375, \beta = 1, \gamma = 0.1$

5 Conclusions

This present analysis exhibits the applicability of the homotopy perturbation Sumudu transform method to solve systems of differential equations of fractional order and the nonlinear biochemical reaction mode. The work emphasized our belief that the method is a reliable technique to handle linear and nonlinear fractional differential equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, restrictive assumptions. The results of this method are in good agreement with those obtained by using the variational iteration method and the Adomian decomposition method. As an advantage of this method over the Adomian decomposition method, in this method we do not need to do the difficult computation for finding the Adomian polynomials, Generally speaking, the proposed method is promising and applicable to a broad class of linear and nonlinear problems in the theory of fractional calculus.

Competing Interests

Authors have declared that no competing interests exist.

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